



# Model of a minimal risk portfolio under hybrid uncertainty

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# Introduction

We investigate the architecture of some models for a minimal risk portfolio under conditions of hybrid uncertainty of possibilistic-probabilistic type.

In our work attention is paid to the study of situations when the interaction of fuzzy model factors is described by both the strongest and weakest t-norms, which allows us to assess the range of risk changes and the behavior of a set of acceptable portfolios, that is, to manage uncertainty when making investment decisions.

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# Necessary mathematical apparatus

**Definition 1.** *Fuzzy random variable  $Y(\omega, \gamma)$  is a real function  $Y : \Omega \times \Gamma \rightarrow E^1$   $\sigma$ -measurable for each fixed  $\gamma$ , where*

$$\mu_Y(\omega, t) = \pi\{\gamma \in \Gamma : Y(\omega, \gamma) = t\}$$

*is called its distribution function.*

It follows from Definition 1 that the distribution function of a fuzzy random variable depends on a random parameter, that is, it is a random function.

**Definition 2.** *Let  $Y(\omega, \gamma)$  be a fuzzy random variable. Its expected value  $E[Y]$  is a fuzzy variable with possibility distribution function*

$$\mu_{E[Y]}(t) = \pi\{\gamma \in \Gamma : E[Y(\omega, \gamma)] = t\},$$

*where  $E$  is the expectation operator*

$$E[Y(\omega, \gamma)] = \int_{\Omega} Y(\omega, \gamma) P(d\omega).$$

# Necessary mathematical apparatus

**Definition 3.** A covariance of fuzzy random variables  $X$  and  $Y$  is defined as:

$$\text{cov}(X, Y) = \frac{1}{2} \int_0^1 \left( \text{cov}(X_\omega^-(\alpha), Y_\omega^-(\alpha)) + \text{cov}(X_\omega^+(\alpha), Y_\omega^+(\alpha)) \right) d\alpha, \quad (1)$$

where  $X_\omega^-(\alpha), Y_\omega^-(\alpha), X_\omega^+(\alpha), Y_\omega^+(\alpha)$  are left and right boundaries of  $\alpha$ -level sets of fuzzy variables  $X_\omega$  and  $Y_\omega$ .

**Definition 4.** A variance of a fuzzy random variable  $Y$  is

$$\mathbf{D}[Y] = \text{cov}(Y, Y).$$

**Definition 5.**  $Z(\gamma)$  is called an LR-type fuzzy variable, if its distribution function has the form

$$\mu_Z(t) = \begin{cases} L\left(\frac{m-t}{d}\right), & \text{for } t < \underline{m}, \\ 1, & \text{for } \underline{m} \leq t \leq \bar{m}, \\ R\left(\frac{t-\bar{m}}{d}\right), & \text{for } t > \bar{m}. \end{cases}$$

where  $L(t)$  and  $R(t)$  are shape functions.

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# Necessary mathematical apparatus

We will use triangular norms (t-norms) to aggregate fuzzy information. These norms generalize "min" operation inherent in operations on fuzzy sets and fuzzy variables (see Nguyen et al. (1997)). The following t-norms are of particular interest:

$$T_M(x, y) = \min\{x, y\} \text{ and } T_W(x, y) = \begin{cases} \min\{x, y\}, & \text{if } \max\{x, y\} = 1, \\ 0, & \text{else,} \end{cases}$$

$T_M$  is called the strongest, and  $T_W$  is called the weakest t-norm, since for any arbitrary t-norm  $T$  and  $\forall x, y \in [0,1]$ , the following inequality holds (see, for example, Nguyen et al. (1997)):

$$T_W(x, y) \leq T(x, y) \leq T_M(x, y).$$

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# Portfolio return under hybrid uncertainty of possibilistic-probabilistic type

Under conditions of hybrid uncertainty of possibilistic-probabilistic type, the return on an investment portfolio can be represented by a fuzzy random function

$$R_p(\boldsymbol{w}, \omega, \gamma) = \sum_{i=1}^n R_i(\omega, \gamma) w_i, \quad (2)$$

which is a linear function of equity shares  $\boldsymbol{w} = (w_1, \dots, w_n)$  in the portfolio. Here  $R_i(\omega, \gamma)$  are fuzzy random variables that model the returns of individual financial assets with the help of shift-scale representation (see Yazenin (2016)):

$$R_i(\omega, \gamma) = a_i(\omega) + \sigma_i(\omega) Z_i(\gamma). \quad (3)$$

# Expected Portfolio Return Under Conditions of Hybrid Uncertainty

Further we assume that fuzzy variables  $Z_i(\gamma) = [\underline{m}_i, \bar{m}_i, \underline{d}_i, \bar{d}_i]_{LR}$  in representation (3) are mutually T-related, where  $T \in \{T_M, T_W\}$ , and  $a_i(\omega), \sigma_i(\omega)$  are shift and scale coefficients – random variables defined on a probability space  $(\Omega, \mathbf{B}, \mathbf{P})$ , with  $\sigma_i(\omega) \geq 0$ . Then possibilities distribution of the portfolio return (2) takes the following form

$$R_p^T(\omega, \omega, \gamma) = \left[ \underline{m}_{R_p}(\omega, \omega), \bar{m}_{R_p}(\omega, \omega), \underline{d}_{R_p^T}(\omega, \omega), \bar{d}_{R_p^T}(\omega, \omega) \right]_{LR}, \quad (4)$$

where

$$\underline{m}_{R_p}(\omega, \omega) = \sum_{i=1}^n (a_i(\omega) + \sigma_i(\omega)\underline{m}_i)\omega_i, \quad \bar{m}_{R_p}(\omega, \omega) = \sum_{i=1}^n (a_i(\omega) + \sigma_i(\omega)\bar{m}_i)\omega_i,$$

and the coefficients of fuzziness take the form depending on the type of T:

$$\underline{d}_{R_p^M}(\omega, \omega) = \sum_{i=1}^n \sigma_i(\omega)\underline{d}_i\omega_i, \quad \bar{d}_{R_p^M}(\omega, \omega) = \sum_{i=1}^n \sigma_i(\omega)\bar{d}_i\omega_i,$$

when  $T = T_M$ , and in case of  $T = T_W$ :

$$\underline{d}_{R_p^W}(\omega, \omega) = \max_{i=1\dots n} \{\sigma_i(\omega)\underline{d}_i\omega_i\}, \quad \bar{d}_{R_p^W}(\omega, \omega) = \max_{i=1\dots n} \{\sigma_i(\omega)\bar{d}_i\omega_i\}.$$

# Expected Portfolio Return Under Conditions of Hybrid Uncertainty

**Theorem 1.** Let  $T = T_M$ . Then expected portfolio return  $\hat{R}_p^M(\boldsymbol{w}, \gamma)$  is characterized by the possibilities distribution function

$$\hat{R}_p^M(\boldsymbol{w}, \gamma) = \mathbf{E}[R_p^M(\boldsymbol{w}, \omega, \gamma)] = \left[ \underline{m}_{\hat{R}_p}(\boldsymbol{w}), \bar{m}_{\hat{R}_p}(\boldsymbol{w}), \underline{d}_{\hat{R}_p^M}(\boldsymbol{w}), \bar{d}_{\hat{R}_p^M}(\boldsymbol{w}) \right]_{LR},$$

where

$$\begin{aligned} \underline{m}_{\hat{R}_p}(\boldsymbol{w}) &= \sum_{i=1}^n (\hat{a}_i + \hat{\sigma}_i \underline{m}_i) w_i, & \bar{m}_{\hat{R}_p}(\boldsymbol{w}) &= \sum_{i=1}^n (\hat{a}_i + \hat{\sigma}_i \bar{m}_i) w_i, \\ \underline{d}_{\hat{R}_p^M}(\boldsymbol{w}) &= \sum_{i=1}^n \hat{\sigma}_i \underline{d}_i w_i, & \bar{d}_{\hat{R}_p^M}(\boldsymbol{w}) &= \sum_{i=1}^n \hat{\sigma}_i \bar{d}_i w_i, & \hat{a}_i &= \mathbf{E}[a_i(\omega)], & \hat{\sigma}_i &= \mathbf{E}[\sigma_i(\omega)]. \end{aligned}$$

**Theorem 2.** Let  $T = T_W$ . Then expected portfolio return  $\hat{R}_p^W(\boldsymbol{w}, \gamma)$  is characterized by the possibilities distribution function

$$\hat{R}_p^W(\boldsymbol{w}, \gamma) = \mathbf{E}[R_p^W(\boldsymbol{w}, \omega, \gamma)] = \left[ \underline{m}_{\hat{R}_p}(\boldsymbol{w}), \bar{m}_{\hat{R}_p}(\boldsymbol{w}), \underline{d}_{\hat{R}_p^W}(\boldsymbol{w}), \bar{d}_{\hat{R}_p^W}(\boldsymbol{w}) \right]_{LR},$$

where

$$\underline{d}_{\hat{R}_p^W}(\boldsymbol{w}) = \mathbf{E} \left[ \max_{i=1 \dots n} \{ \sigma_i(\omega) \underline{d}_i w_i \} \right], \quad \bar{d}_{\hat{R}_p^W}(\boldsymbol{w}) = \mathbf{E} \left[ \max_{i=1 \dots n} \{ \sigma_i(\omega) \bar{d}_i w_i \} \right].$$



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# Models of acceptable portfolios under hybrid uncertainty of possibilistic-probabilistic type

In accordance with the classical Markowitz (1952) approach, we need to construct a portfolio risk function in the minimal risk portfolio model.

$$F_p^{\tau E}(w) = \begin{cases} \tau\{\hat{R}_p^T(w, \gamma) \mathcal{R} m_d\} \geq \alpha, \\ \sum_{i=1}^n w_i = 1, \\ w \in \mathbb{E}_+^n, \end{cases}$$

where  $\mathbb{E}_+^n = \{x \in \mathbb{E}^n : x \geq 0\}$ ,  $\hat{R}_p^T(w, \gamma)$  – expected return,  $\mathcal{R}$  – crisp relation  $\{\geq, =\}$ ;  $\alpha \in (0, 1]$ ,  $m_d$  – level of profitableness, acceptable to an investor,  $T \in \{T_M, T_W\}$ .

# Models of acceptable portfolios under hybrid uncertainty of possibilistic-probabilistic type

The following theorems allow us to construct equivalent deterministic analogs of acceptable portfolio models.

**Theorem 3.** *Let in the constraint model  $F_p^{\tau E}$   $\tau = ' \pi '$ ,  $\mathcal{R} = ' \geq '$ . Then the equivalent deterministic model of acceptable portfolios has the form:*

$$F_p^{\pi E}(\omega) = \begin{cases} \sum_{i=1}^n (\hat{a}_i + \hat{\sigma}_i \bar{m}_i) \omega_i + \bar{d}_{\hat{R}_p^T}(\omega) * R^{-1}(\alpha) \geq m_d, \\ \sum_{i=1}^n \omega_i = 1, \\ \omega \in \mathbb{E}_+^n. \end{cases}$$

# Models of acceptable portfolios under hybrid uncertainty of possibilistic-probabilistic type

**Theorem 4.** *Let in the model of acceptable portfolios  $F_p^{\tau E}$   $\tau = 'v'$ ,  $\mathcal{R} = '\geq'$ . Then the equivalent deterministic model of acceptable portfolios takes the form:*

$$F_p^{vE}(\omega) = \begin{cases} \sum_{i=1}^n (\hat{a}_i + \hat{\sigma}_i \underline{m}_i) \omega_i - \underline{d}_{\hat{R}_p^T}(\omega) * L^{-1}(1 - \alpha) \geq m_d, \\ \sum_{i=1}^n \omega_i = 1, \\ \omega \in \mathbb{E}_+^n. \end{cases}$$

From theorems 3, 4 we get

**Corollary 1.**  $F_p^{vE}(\omega) \subseteq F_p^{\pi E}(\omega)$ .

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## Models of acceptable portfolios under hybrid uncertainty of possibilistic-probabilistic type

In case when portfolio return (2) is included in the system of restrictions, the hybrid uncertainty can be removed by imposing a limit on the possibility/necessity and probability of an acceptable level of return.

Formally, the mathematical model of such a constraint can be written as:

$$F_p^{\tau\mathbf{P}}(\omega) = \begin{cases} \tau\{\mathbf{P}\{R_p(\omega, \omega, \gamma) \mathcal{R} m_d\} \geq p_0\} \geq \alpha_0, \\ \sum_{i=1}^n \omega_i = 1, \\ \omega \in \mathbb{E}_+^n, \end{cases}$$

where  $\mathbf{P}$  – probability measure,  $p_0 \in (0,1]$  – probability level.

# Models of acceptable portfolios under hybrid uncertainty of possibilistic-probabilistic type

For further analysis we will need the following notation and concepts. We denote by  $t = (t_1, \dots, t_n)$  – a vector whose components are possible values of fuzzy variables  $Z_1(\gamma), \dots, Z_n(\gamma)$  respectively. For the strongest t-norm with the possibility of  $\mu_{Z_i}(t_i)$ , the return of the i-th financial asset is a random variable  $Z_i^{t_i}(\omega) = a_i(\omega) + \sigma_i(\omega)t_i$ , and  $R_p^t(\omega, \omega) = \sum_{i=1}^n (a_i(\omega) + \sigma_i(\omega)t_i)\omega_i$  is return on the portfolio with the possibility of  $\mu_p(t) = \min_{1 \leq i \leq n} \{\mu_{Z_i}(t_i)\}$ .

Then, following Yazenin (2016), with the possibility of  $\mu_p(t)$ , the expected return and risk of the portfolio are determined by the formulas

$$m_{R_p}(\omega, t) = \mathbf{E}[R_p^t(\omega, \omega)] = \sum_{i=1}^n (\hat{a}_i + \hat{\sigma}_i t_i)\omega_i$$

and

$$d_{R_p}(\omega, t) = \mathbf{E} \left[ \left( R_p^t(\omega, \omega) - m_{R_p}(\omega, t) \right)^2 \right].$$

# Models of acceptable portfolios under hybrid uncertainty of possibilistic-probabilistic type

Using standard transformations we obtain the following formula for the variance with the possibility of  $\mu_p(t)$  (see, for example, Yazenin (2016)):

$$d_{R_p}(w, t) = \sum_{i=1}^n \sum_{j=1}^n c_{ij}(t_i, t_j) w_i w_j,$$

in which

$$c_{ij}(t_i, t_j) = C_{a_i a_j} + C_{a_j \sigma_i} t_j + C_{\sigma_j a_i} t_i + C_{\sigma_i \sigma_j} t_i t_j,$$

$$C_{a_i a_j} = \text{cov}(a_i, a_j), C_{a_j \sigma_i} = \text{cov}(a_j, \sigma_i), C_{\sigma_j a_i} = \text{cov}(\sigma_j, a_i), C_{\sigma_i \sigma_j} = \text{cov}(\sigma_i, \sigma_j).$$

Function  $d_{R_p}(w, t)$  has properties that are due to the properties of the covariance matrix  $\mathbf{C}$  with elements  $c_{ij}(t_i, t_j)$ :

- $d_{R_p}(w, t)$  is a convex function on  $w$  for a fixed  $t$ ;
- for any vectors  $w$  and  $t$  function  $d_{R_p}(w, t)$  is nonnegative;
- $d_{R_p}(w, t)$  is a convex function on  $t$  for a fixed  $w$ .

The function  $d_{R_p}(w, t)$  can be written with the help of matrix  $\mathbf{C}$  as

$$d_{R_p}(w, t) = (\mathbf{C} w, w).$$

# Models of acceptable portfolios under hybrid uncertainty of possibilistic-probabilistic type

**Lemma 1.** *Let in the constraint model  $F_p^{\tau P}(\omega)$  random parameters are normally distributed:  $a_i(\omega) \in \mathcal{N}_p(\hat{a}_i, \hat{d}_{a_i})$ ,  $\sigma_i(\omega) \in \mathcal{N}_p(\hat{\sigma}_i, \hat{d}_{\sigma_i})$ ,  $i = 1, \dots, n$ ; fuzzy parameters  $Z_1(\gamma), \dots, Z_N(\gamma)$  are  $T_M$ -related,  $\mu_p(t) \geq \alpha_0$ . Then with the possibility of  $\mu_p(t)$  system of restrictions  $F_p^{\tau P}(\omega)$  is equivalent to the system*

$$F_p^{\mu P}(\omega) = \begin{cases} m_{R_p}(\omega, t) + \beta_0 \sqrt{d_{R_p}(\omega, t)} \geq m_d, \\ \sum_{i=1}^n \omega_i = 1, \\ \omega \in \mathbb{E}_+^n, \end{cases}$$

where  $\beta_0$  – is a solution to the equation  $\mathcal{F}_0^1(x) = 1 - p_0$ , and  $\mathcal{F}_0^1(x)$  – is a function of the standard normal probabilities distribution.

# Models of acceptable portfolios under hybrid uncertainty of possibilistic-probabilistic type

The Lemma proved above allows us to prove a theorem with the help of which an equivalent deterministic analog of the system  $F_p^{\tau P}(\omega)$  can be constructed.

**Theorem 5.** *Let in the constraint model  $F_p^{\tau P}(\omega)$  random parameters are normally distributed:  $a_i(\omega) \in \mathcal{N}_p(\hat{a}_i, \hat{d}_{a_i})$ ,  $\sigma_i(\omega) \in \mathcal{N}_p(\hat{\sigma}_i, \hat{d}_{\sigma_i})$ ,  $i = 1, \dots, n$ ; fuzzy parameters  $Z_1(\gamma), \dots, Z_N(\gamma)$  are  $T_M$ -related,  $\tau = '\pi'$ . Then the system of restrictions  $F_p^{\tau P}(\omega)$  is equivalent to the system*

$$F_p^{\pi P}(\omega) = \begin{cases} m_{R_p}(\omega, t^+) + \beta_0 \sqrt{d_{R_p}(\omega, t^+)} \geq m_d, \\ \sum_{i=1}^n \omega_i = 1, \\ \omega \in \mathbb{E}_+^n, \end{cases}$$

where  $m_{R_p}(\omega, t^+) = \sum_{i=1}^n (\hat{a}_i + \hat{\sigma}_i t_i^+) \omega_i$ ,  $d_{R_p}(\omega, t^+) = \sum_{i=1}^n \sum_{j=1}^n c_{ij}(t_i^+, t_j^+) \omega_i \omega_j$ , and  $t_i^+, t_j^+$  are right borders of  $\alpha_0$ -level sets of fuzzy variables  $Z_i(\gamma), Z_j(\gamma)$ , respectively.



# Assessment of portfolio risk with hybrid uncertainty

In accordance with the indicated approach to determining second-order moments, we can determine the variance of the portfolio to assess its risk. We need to obtain formulas (variances) for the case of both the strongest and weakest t-norms.

With  $T = T_M$  formula (1) takes the form

$$D_p^M(\omega) = \frac{1}{2} \int_0^1 (\mathbf{D}[R_p^{M-}(\omega, \omega, \alpha)] + \mathbf{D}[R_p^{M+}(\omega, \omega, \alpha)]) d\alpha,$$

where  $R_p^{M-}(\omega, \omega, \alpha)$  and  $R_p^{M+}(\omega, \omega, \alpha)$  – are left and right boundaries of  $\alpha$ -level set of fuzzy random variable  $R_p^M(\omega, \omega, \alpha)$ :

$$R_p^{M-}(\omega, \omega, \alpha) = \sum_{i=1}^n (a_i(\omega) + \sigma_i(\omega)\underline{m}_i)\omega_i - \sum_{i=1}^n \sigma_i(\omega)\underline{d}_i\omega_i * L^{-1}(\alpha),$$
$$R_p^{M+}(\omega, \omega, \alpha) = \sum_{i=1}^n (a_i(\omega) + \sigma_i(\omega)\bar{m}_i)\omega_i + \sum_{i=1}^n \sigma_i(\omega)\bar{d}_i\omega_i * R^{-1}(\alpha).$$

# Assessment of portfolio risk with hybrid uncertainty

If all random parameters of distributions are independent, then after standard transformations we get the formula for the variance:

$$\begin{aligned} & D_p^M(w) \\ &= \sum_{i=1}^n \left( \mathbf{D}[a_i(\omega)] \right. \\ &+ \frac{1}{2} \mathbf{D}[\sigma_i(\omega)] \left( \underline{m}_i^2 + \overline{m}_i^2 + 2\overline{m}_i \overline{d}_i \int_0^1 R^{-1}(\alpha) d\alpha - 2\underline{m}_i \underline{d}_i \int_0^1 L^{-1}(\alpha) d\alpha \right. \\ &\left. \left. + \overline{d}_i^2 \int_0^1 (R^{-1}(\alpha))^2 d\alpha + \underline{d}_i^2 \int_0^1 (L^{-1}(\alpha))^2 d\alpha \right) \right) w_i^2. \end{aligned}$$

# Assessment of portfolio risk with hybrid uncertainty

Note that if in fuzzy random variables the fuzzy components are given by LR-type fuzzy numbers with the same left and right shapes and coefficients of fuzziness, i.e.  $L(\alpha) = R(\alpha) = S(\alpha), \forall \alpha$  and  $\bar{m}_i = \underline{m}_i = m_i, \bar{d}_i = \underline{d}_i = d_i, i = 1, \dots, n$ , then the variance formula can be simplified:

$$D_p^M(w) = \sum_{i=1}^n \left( \mathbf{D}[a_i(\omega)] + \mathbf{D}[\sigma_i(\omega)] \left( m_i^2 + d_i^2 \int_0^1 (S^{-1}(\alpha))^2 d\alpha \right) \right) w_i^2.$$

**Example 1.** In case when shift and scale coefficients  $a_i(\omega), \sigma_i(\omega)$  are uniformly distributed over the segment  $[0,1]$  and independent, and the shape function  $S(t) = \max\{0, 1 - t\}, t \geq 0$ , we obtain the following formula for the variance:

$$D_p^M(w) = \frac{1}{12} \sum_{i=1}^n \left( m_i^2 + \frac{1}{3} d_i^2 + 1 \right) w_i^2.$$

# Assessment of portfolio risk with hybrid uncertainty

We now define the variance for the t-norm  $T_W$ . To do this, we will again use formula (1) to find the covariance of two fuzzy random variables. For the weakest t-norm, the formula for finding the variance takes the form:

$$D_p^W(\omega) = \frac{1}{2} \int_0^1 (\mathbf{D}[R_p^{W-}(\omega, \omega, \alpha)] + \mathbf{D}[R_p^{W+}(\omega, \omega, \alpha)]) d\alpha,$$

where  $R_p^{W-}(\omega, \omega, \alpha)$  and  $R_p^{W+}(\omega, \omega, \alpha)$  are respectively left and right boundaries of  $\alpha$ -level set of portfolio return – fuzzy random variable  $R_p^W(\omega, \omega, \gamma)$ :

$$R_p^{W-}(\omega, \omega, \alpha) = \sum_{i=1}^n (a_i(\omega) + \sigma_i(\omega) \underline{m}_i) \omega_i - \max_{i=1 \dots n} \{ \sigma_i(\omega) \underline{d}_i \omega_i \} * L^{-1}(\alpha),$$

$$R_p^{W+}(\omega, \omega, \alpha) = \sum_{i=1}^n (a_i(\omega) + \sigma_i(\omega) \bar{m}_i) \omega_i + \max_{i=1 \dots n} \{ \sigma_i(\omega) \bar{d}_i \omega_i \} * R^{-1}(\alpha).$$

# Assessment of portfolio risk with hybrid uncertainty

After appropriate transformations, the final formula for variance, which allows us to determine the risk of the portfolio, has the form:

$$\begin{aligned}
 & D_p^W(\omega) \\
 &= \frac{1}{2} \sum_{i=1}^n w_i^2 (\mathbf{D}[a_i(\omega) + \sigma_i(\omega)\underline{m}_i] + \mathbf{D}[a_i(\omega) + \sigma_i(\omega)\overline{m}_i]) \\
 &+ \frac{1}{2} \mathbf{D} \left[ \max_{j=1\dots n} \{\sigma_j(\omega)\overline{d}_j w_j\} \right] \int_0^1 (R^{-1}(\alpha))^2 d\alpha + \frac{1}{2} \mathbf{D} \left[ \max_{j=1\dots n} \{\sigma_j(\omega)\underline{d}_j w_j\} \right] \int_0^1 (L^{-1}(\alpha))^2 d\alpha \\
 &+ \sum_{1 \leq i < j \leq n} w_i w_j \left( \text{cov} \left( (a_i(\omega) + \sigma_i(\omega)\underline{m}_i), (a_j(\omega) + \sigma_j(\omega)\underline{m}_j) \right) \right. \\
 &+ \left. \text{cov} \left( (a_i(\omega) + \sigma_i(\omega)\overline{m}_i), (a_j(\omega) + \sigma_j(\omega)\overline{m}_j) \right) \right) \\
 &+ \sum_{i=1}^n w_i \left( \int_0^1 R^{-1}(\alpha) d\alpha \text{cov} \left( (a_i(\omega) + \sigma_i(\omega)\overline{m}_i), \max_{j=1\dots n} \{\sigma_j(\omega)\overline{d}_j w_j\} \right) \right. \\
 &\left. - \int_0^1 L^{-1}(\alpha) d\alpha \text{cov} \left( (a_i(\omega) + \sigma_i(\omega)\underline{m}_i), \max_{j=1\dots n} \{\sigma_j(\omega)\underline{d}_j w_j\} \right) \right).
 \end{aligned}$$

# Assessment of portfolio risk with hybrid uncertainty

If all random parameters of distributions are independent, then:

$$\begin{aligned}
 & D_p^W(\boldsymbol{w}) \\
 &= \frac{1}{2} \sum_{i=1}^n w_i^2 \left( 2\mathbf{D}[a_i(\omega)] + \mathbf{D}[\sigma_i(\omega)] (\underline{m}_i^2 + \overline{m}_i^2) \right) \\
 &+ \frac{1}{2} \mathbf{D} \left[ \max_{j=1\dots n} \{ \sigma_j(\omega) \overline{d}_j w_j \} \right] \int_0^1 (R^{-1}(\alpha))^2 d\alpha + \frac{1}{2} \mathbf{D} \left[ \max_{j=1\dots n} \{ \sigma_j(\omega) \underline{d}_j w_j \} \right] \int_0^1 (L^{-1}(\alpha))^2 d\alpha \\
 &+ \sum_{i=1}^n w_i \left( \int_0^1 R^{-1}(\alpha) d\alpha \operatorname{cov} \left( (a_i(\omega) + \sigma_i(\omega) \overline{m}_i), \max_{j=1\dots n} \{ \sigma_j(\omega) \overline{d}_j w_j \} \right) \right. \\
 &\left. - \int_0^1 L^{-1}(\alpha) d\alpha \operatorname{cov} \left( (a_i(\omega) + \sigma_i(\omega) \underline{m}_i), \max_{j=1\dots n} \{ \sigma_j(\omega) \underline{d}_j w_j \} \right) \right).
 \end{aligned}$$

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# Assessment of portfolio risk with hybrid uncertainty

If in all distributions the fuzzy component is given by LR-type fuzzy numbers with the same left and right shape functions and coefficients of fuzziness, then the terms with covariance are mutually eliminated and the variance formula can be simplified:

$$\begin{aligned} D_p^W(w) &= \sum_{i=1}^n w_i^2 (\mathbf{D}[a_i(\omega)] + \mathbf{D}[\sigma_i(\omega)]m_i^2) \\ &+ \mathbf{D} \left[ \max_{j=1 \dots n} \{ \sigma_j(\omega) d_j w_j \} \right] \int_0^1 (S^{-1}(\alpha))^2 d\alpha. \end{aligned} \quad (5)$$

# Assessment of portfolio risk with hybrid uncertainty

**Example 2.** Let the shift and scale coefficients  $a_i(\omega), \sigma_i(\omega)$  are uniformly distributed over the segment  $[0,1]$  and independent the formula (5) under the corresponding assumptions and for  $S(t) = \max\{0, 1 - t\}, t \geq 0$  takes the form:

$$D_p^W(w) = \frac{1}{12} \sum_{i=1}^n w_i^2 (m_i^2 + 1) + \frac{1}{3} \left( EMax2(dw) - (EMax(dw))^2 \right),$$

where

$$\begin{aligned} EMax(dw) &:= \mathbf{E} \left[ \max_{i=1 \dots n} \{ \sigma_i(\omega) d_i w_i \} \right] \\ &= \sum_{i=1}^n \frac{(dw)_{(i)}^{n-i+1}}{(n-i+1)(n-i+2)(dw)_{(i+1)} \cdots (dw)_{(n)}}, \end{aligned}$$

$$\begin{aligned} EMax2(dw) &:= \mathbf{E} \left[ \left( \max_{i=1 \dots n} \{ \sigma_i(\omega) d_i w_i \} \right)^2 \right] \\ &= \sum_{i=1}^n \frac{2(dw)_{(i)}^{n-i+2}}{(n-i+2)(n-i+3)(dw)_{(i+1)} \cdots (dw)_{(n)}}, \end{aligned}$$

and  $(dw)_{(1)}, \dots, (dw)_{(n)}$  is an ascending permutation of elements  $\{d_1 w_1, \dots, d_n w_n\}$ .



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# Minimum Risk Portfolio Models

Based on the results presented above, the minimum risk portfolio models can be written as:

$$D_p^T(w) \rightarrow \min, \quad (6)$$

$$w \in F_p(w), \quad (7)$$

where  $F_p(w) \in \{F_p^{\mu E}, F_p^{\nu E}, F_p^{\mu P}, F_p^{\nu P}\}$  and  $T \in \{M, W\}$ .

# Numerical Calculations

We consider an example of two-dimensional portfolio ( $n = 2$ ). Let  $Z_1 = [2.2, 2.2, 0.3, 0.3]_{LR}$ ,  $Z_2 = [1.2, 1.2, 0.4, 0.4]_{LR}$ ,  $L(t) = R(t) = \max\{0, 1 - t\}$ ,  $t \geq 0$ ,  $\alpha = 0.75$ . Recall that all  $a_i(\omega), \sigma_i(\omega)$  are independent random variables with a uniform distribution on the segment  $[0,1]$ . We first specify the minimum risk portfolio models for the weakest t-norm:

$$\frac{73}{150}w_1^2 + \frac{61}{300}w_2^2 + \frac{1}{3}\left(EMax2(dw) - (EMax(dw))^2\right) \rightarrow \min,$$
$$F_p^{\pi E}(w) = \begin{cases} 1.6w_1 + 1.1w_2 + 0.25 * EMax(dw) \geq m_d, \\ w_1 + w_2 = 1, \\ w_1, w_2 \geq 0, \end{cases}$$

and in the context of a necessity measure:

$$\frac{73}{150}w_1^2 + \frac{61}{300}w_2^2 + \frac{1}{3}\left(EMax2(dw) - (EMax(dw))^2\right) \rightarrow \min,$$
$$F_p^{\nu E}(w) = \begin{cases} 1.6w_1 + 1.1w_2 - 0.75 * EMax(dw) \geq m_d, \\ w_1 + w_2 = 1, \\ w_1, w_2 \geq 0. \end{cases}$$

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# Numerical Calculations

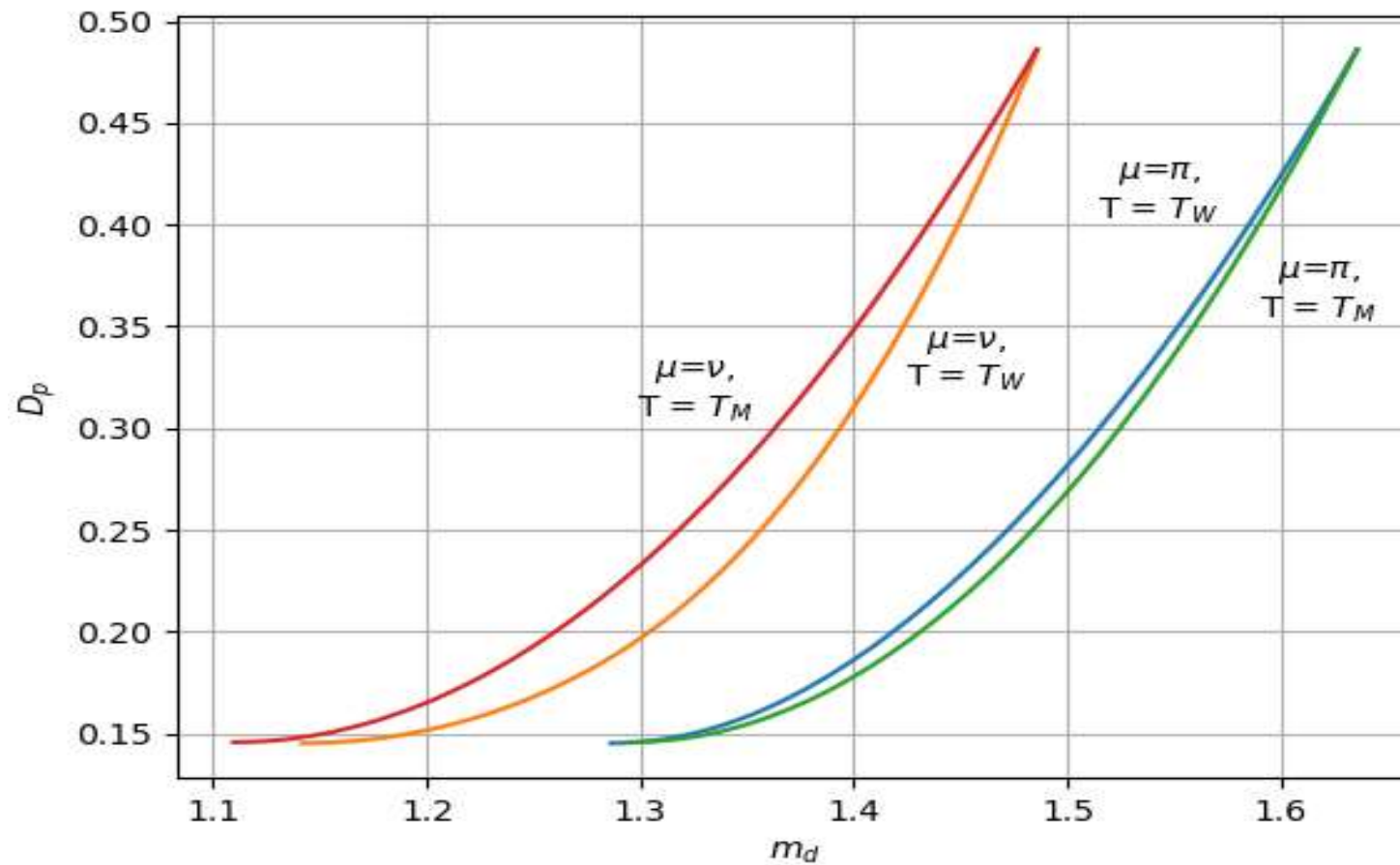
Let us now consider the same problem for the strongest t-norm. Under the assumptions made, the equivalent deterministic analog of the minimum risk portfolio (6)-(7) in the context of the possibility measure takes the form:

$$F_p^{\pi E}(w) = \begin{cases} \frac{587}{1200} w_1^2 + \frac{187}{900} w_2^2 \rightarrow \min, \\ 1.6375 w_1 + 1.15 w_2 \geq m_d, \\ w_1 + w_2 = 1, \\ w_1, w_2 \geq 0, \end{cases}$$

and in the context of a necessity measure:

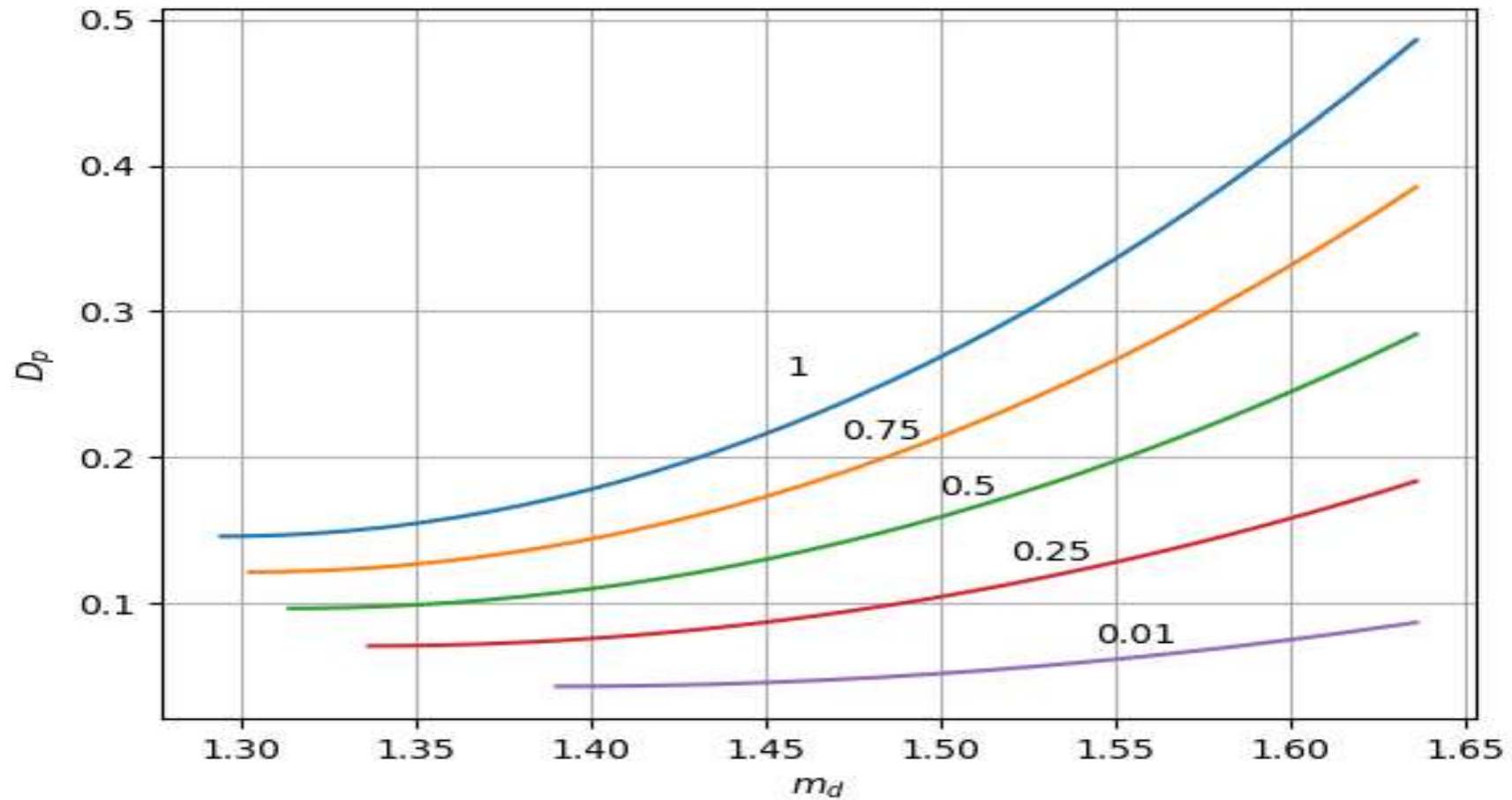
$$F_p^{\nu E}(w) = \begin{cases} \frac{587}{1200} w_1^2 + \frac{187}{900} w_2^2 \rightarrow \min, \\ 1.4875 w_1 + 0.95 w_2 \geq m_d, \\ w_1 + w_2 = 1, \\ w_1, w_2 \geq 0. \end{cases}$$

# Numerical Calculations



*Fig. 1: Sets of quasi-efficient portfolios depending on the measure of possibility/necessity and the t-norm*

# Numerical Calculations



*Fig. 2: Influence of the random parameter  $\sigma_i$  (scaling) on the set of quasi-efficient portfolios in the case of the strongest  $t$ -norm in the context of the measure of possibility*

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# Thank you!