

A Multiobjective Proximal Bundle Method for Nonsmooth Invex Fuzzy Optimization

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Abstract. We study a multiobjective proximal bundle framework for nonsmooth fuzzy-valued optimization problems whose deterministic endpoint functions satisfy generalized invexity assumptions. The fuzzy objective is represented through its α -cuts and is thereby reduced, for each $\alpha \in [0, 1]$, to a crisp biobjective problem involving the left and right endpoints. This reformulation makes it possible to combine Clarke nonsmooth analysis, generalized invexity and a proximal bundle method in a unified setting. We first collect the nonsmooth and fuzzy notions required for a self-contained treatment, including Clarke's generalized directional derivative and subdifferential, pseudo- and quasiinvexity, weak Pareto optimality, fuzzy numbers and α -cut representations. We then establish an optimality principle based on a max-type improvement function and formulate a multiobjective proximal bundle method with bundle linearizations, locality measures, proximal stabilization and a two-point line search. Under a common invexity mapping, a constraint qualification and weak semismoothness, finite termination yields a global weak Pareto solution and every accumulation point of an infinite sequence has the same property. The fuzzy specialization is illustrated numerically on a nonsmooth constrained problem with a triangular fuzzy parameter.

Keywords: nonsmooth optimization, fuzzy optimization, invexity, multiobjective optimization, proximal bundle method, Clarke subdifferential

1 Introduction

Nonsmooth optimization, multiobjective decision making and uncertainty frequently occur simultaneously in applications. Bundle methods are among the standard numerical tools for locally Lipschitz and other nonsmooth optimization problems because they retain subgradient information from previous iterations and stabilize the resulting cutting-plane model by a proximal term; see, e.g., [11, 13, 14, 2]. Multiobjective bundle procedures extend this idea by seeking directions that improve all objective components in a suitable max-type sense [12, 20, 30, 16].

A second ingredient of the present work is generalized convexity. Invexity was introduced as a framework in which important global optimality properties of convex programming remain valid beyond the convex class [9, 6, 7,

17]. Nonsmooth variants based on generalized subdifferentials were developed subsequently; in particular, Reiland's formulation provides a natural Clarke-subdifferential version of invexity [25]. Pseudo- and quasiinvexity are especially convenient for vector optimization because they can convert stationarity-type conditions into global conclusions [22, 15]. Care is nevertheless required when several functions are assumed invex with respect to a common mapping η ; this issue is central in constrained and multiobjective results and has been discussed critically in the literature [18, 33].

Uncertainty can be represented by fuzzy numbers when the available information is better interpreted in terms of degrees of possibility than through a probability law. Fuzzy mathematical programming goes back to the seminal work of Bellman and Zadeh and Tanaka and coauthors [3, 29]. For fuzzy-valued objective functions, α -cuts provide a particularly useful bridge to deterministic optimization: a fuzzy value is represented by an interval and the left and right endpoints become two real-valued objectives [23, 31, 32, 26, 4, 24, 1].

The purpose of this paper is to give a self-contained development of a multiobjective proximal bundle method (MPBM) for this setting. The main contributions are as follows. First, the necessary concepts from Clarke nonsmooth analysis, generalized invexity, vector optimization and fuzzy optimization are assembled in a common notation. Second, an improvement function is used to connect a constrained multiobjective problem with a scalar nonsmooth minimization model. Third, a proximal bundle subproblem, a two-point line search and an aggregation/update mechanism are presented in a form directly applicable to the two endpoint functions generated by a fuzzy objective. Finally, a global convergence conclusion is obtained under pseudoinvexity, weak semismoothness and a standard constraint qualification. The development extends the multiobjective bundle construction of [16] from generalized convexity assumptions to the common- η invex setting and specializes it to fuzzy-valued optimization.

2 Nonsmooth and generalized-invex preliminaries

Throughout, $\|\cdot\|$ denotes the Euclidean norm and $\langle \cdot, \cdot \rangle$ the Euclidean inner product. All functions considered in the nonsmooth part are real-valued and locally Lipschitz continuous on an open set containing the points of interest unless stated otherwise.

Definition 1 (Local Lipschitz continuity). *A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is locally Lipschitz continuous at $x \in \mathbb{R}^n$ if there exist $K_x > 0$ and $\delta_x > 0$ such that*

$$|f(y) - f(z)| \leq K_x \|y - z\| \quad \text{for all } y, z \in B(x; \delta_x).$$

For a locally Lipschitz function, Clarke's generalized directional derivative at x in direction d is [5]

$$f^\circ(x; d) = \limsup_{\substack{y \rightarrow x \\ t \downarrow 0}} \frac{f(y + td) - f(y)}{t}. \quad (1)$$

The associated Clarke subdifferential is

$$\partial f(x) = \{\xi \in \mathbb{R}^n : f^\circ(x; d) \geq \langle \xi, d \rangle \text{ for all } d \in \mathbb{R}^n\}. \quad (2)$$

It is a nonempty compact convex set and satisfies the support-function identity

$$f^\circ(x; d) = \max_{\xi \in \partial f(x)} \langle \xi, d \rangle. \quad (3)$$

Moreover, if f has a local minimum at x , then $0 \in \partial f(x)$ [5, 2].

We use the conic hull notation

$$\text{cone } S := \{\lambda z : \lambda \geq 0, z \in \text{conv } S\}.$$

For a family of locally Lipschitz functions, generalized invexity will always refer to one and the same mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ whenever a common η is explicitly assumed.

Definition 2 (Invexity). *A locally Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is invex with respect to η if*

$$f(x) - f(u) \geq \langle \xi, \eta(x, u) \rangle \quad \text{for all } x, u \in \mathbb{R}^n, \quad \xi \in \partial f(u). \quad (4)$$

Equivalently, by (3), $f(x) - f(u) \geq f^\circ(u; \eta(x, u))$.

This Clarke-subdifferential definition is the nonsmooth analogue used in [25]; the differentiable roots of invexity can be found in [9, 6].

Definition 3 (Pseudoinvexity and quasiinvexity). *A locally Lipschitz function f is pseudoinvex with respect to η if*

$$f^\circ(u; \eta(x, u)) \geq 0 \implies f(x) \geq f(u) \quad (5)$$

for all x, u . It is quasiinvex with respect to η if

$$f(x) \leq f(u) \implies f^\circ(u; \eta(x, u)) \leq 0 \quad (6)$$

for all x, u .

The implication in (5) makes pseudoinvexity particularly useful in global optimization: stationary points are global minimizers under the usual regularity assumptions; compare [25, 22]. In the multiobjective setting below, the same η is required for all component functions appearing in the max model. This common- η requirement is not merely notational; without it, summation and maximum arguments used in the optimality and convergence proofs need not remain valid [18, 33].

Definition 4 (Weak semismoothness). *A locally Lipschitz function f is weakly semismooth at x if for every $d \in \mathbb{R}^n$ the limit*

$$\lim_{\substack{t \rightarrow 0 \\ \xi \in \partial f(x+td)}} \langle \xi, d \rangle$$

exists. The function is weakly semismooth if this property holds at every point under consideration.

Weak semismoothness is a standard regularity hypothesis in the convergence analysis of bundle-type methods; see [21, 28, 16].

3 Vector and fuzzy-valued optimization

Consider the constrained vector minimization problem

$$\begin{aligned} & \text{minimize} && F(x) = (f_1(x), \dots, f_p(x))^\top, \\ & \text{subject to} && g_j(x) \leq 0, \quad j = 1, \dots, m, \\ & && x \in \mathbb{R}^n, \end{aligned} \tag{7}$$

with feasible set X . Standard background on vector optimization is given in [27, 19, 10, 8].

Definition 5 (Pareto solutions). *A feasible point $x^* \in X$ is Pareto optimal if there is no $x \in X$ satisfying $f_i(x) \leq f_i(x^*)$ for all i and a strict inequality for at least one component. It is weakly Pareto optimal if there is no $x \in X$ such that $f_i(x) < f_i(x^*)$ for every $i = 1, \dots, p$.*

We now recall the fuzzy notions needed to turn a fuzzy objective into a vector objective. Let $\mathcal{F}(\mathbb{R})$ denote the family of fuzzy intervals on $\mathbb{R}$.

Definition 6 (Fuzzy set and α -cut). *A fuzzy set on $S \subset \mathbb{R}^n$ is a mapping $\tilde{u} : S \rightarrow [0, 1]$. For $\alpha \in (0, 1]$, its α -level set is*

$$[\tilde{u}]_\alpha = \{x \in \mathbb{R}^n : \tilde{u}(x) \geq \alpha\}.$$

For $\alpha = 0$, $[\tilde{u}]_0$ is understood as the closure of the support.

A fuzzy number is a normal, upper semicontinuous fuzzy interval whose 1-level set is a singleton. Its α -cut is a compact interval

$$[\tilde{u}]_\alpha = [u^L(\alpha), u^R(\alpha)], \quad \alpha \in [0, 1]. \tag{8}$$

In particular, for a triangular fuzzy number $\tilde{u} = (u_1, u_2, u_3)$,

$$[\tilde{u}]_\alpha = [(1 - \alpha)u_1 + \alpha u_2, (1 - \alpha)u_3 + \alpha u_2]. \tag{9}$$

These standard constructions can be found, for example, in [23, 31, 32, 1].

Definition 7 (Fuzzy-valued mapping). *Let $D \subset \mathbb{R}^n$ be nonempty. A fuzzy-valued mapping $\tilde{f} : D \rightarrow \mathcal{F}(\mathbb{R})$ has, for each $\alpha \in [0, 1]$, the interval representation*

$$[\tilde{f}(x)]_\alpha = [f^L(x, \alpha), f^R(x, \alpha)], \tag{10}$$

where $f^L(\cdot, \alpha)$ and $f^R(\cdot, \alpha)$ are the left and right endpoint functions.

We adopt the componentwise interval order used in fuzzy optimization [32, 1]. At a fixed α , the fuzzy minimization problem therefore induces the biobjective problem

$$(\text{VMP}_\alpha) \quad \text{minimize}_{x \in D} (f^L(x, \alpha), f^R(x, \alpha)). \tag{11}$$

Thus a nonsmooth fuzzy problem can be handled level-wise by ordinary vector-optimization tools. Relations between fuzzy nondominance and the Pareto solutions of the associated endpoint problems are discussed in [26, 4, 24, 1].

Remark 1. The algorithm developed below is applied to (11) for one or several values of α . If the same feasible point is Pareto optimal for all $\alpha \in [0, 1]$ under the adopted endpoint order, then no feasible fuzzy value can dominate it simultaneously on all α -cuts; hence it is an F-nondominated solution in this ordering. In computation, a finite grid of α -levels gives an approximation to this level-wise requirement.

4 An improvement function and a global optimality principle

For (7), define the active index set

$$J(x) = \{j \in \{1, \dots, m\} : g_j(x) = 0\},$$

and the sets

$$\mathcal{F}(x) = \bigcup_{i=1}^p \partial f_i(x), \quad \mathcal{G}(x) = \bigcup_{j \in J(x)} \partial g_j(x). \quad (12)$$

We use the constraint qualification

$$0 \notin \text{conv } \mathcal{G}(x). \quad (13)$$

The following max-stability observation is the key to treating all objective components in one scalar bundle model.

Theorem 1. *Let $\phi_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, \dots, q$, be locally Lipschitz and pseudoinvex with respect to the same mapping η . Then*

$$\phi(x) = \max_{1 \leq i \leq q} \phi_i(x)$$

is pseudoinvex with respect to η .

Proof. Assume $\phi(y) < \phi(x)$ and let $I(x) = \{i : \phi_i(x) = \phi(x)\}$. Then for every $i \in I(x)$, $\phi_i(y) < \phi_i(x)$, so pseudoinvexity gives $\phi_i^\circ(x; \eta(y, x)) < 0$. For a finite maximum of locally Lipschitz functions,

$$\partial \phi(x) \subset \text{conv} \bigcup_{i \in I(x)} \partial \phi_i(x).$$

Using (3), every element of the right-hand convex hull has negative inner product with $\eta(y, x)$, and therefore $\phi^\circ(x; \eta(y, x)) < 0$. This is the contrapositive form of pseudoinvexity.

The stationarity condition used by the method has a direct global interpretation.

Theorem 2. *Let u be feasible for (7). Suppose that every f_i is pseudoinvex with respect to the same η , every g_j is quasinvex with respect to the same η , and*

$$0 \in \text{conv } \mathcal{F}(u) + \text{cone } \mathcal{G}(u). \quad (14)$$

Then u is a global weakly Pareto optimal solution of (7).

Proof. Let x be any feasible point. For $j \in J(u)$ we have $g_j(x) \leq g_j(u) = 0$. Quasinvexity implies $g_j^\circ(u; \eta(x, u)) \leq 0$. From (14), there exist $\xi_i \in \partial f_i(u)$, $\zeta_j \in \partial g_j(u)$ and nonnegative coefficients λ_i, μ_j with $\sum_i \lambda_i = 1$ such that

$$0 = \sum_{i=1}^p \lambda_i \xi_i + \sum_{j \in J(u)} \mu_j \zeta_j.$$

Taking the inner product with $\eta(x, u)$ and using (3) gives

$$0 \leq \sum_{i=1}^p \lambda_i f_i^\circ(u; \eta(x, u)).$$

Hence at least one component satisfies $f_i^\circ(u; \eta(x, u)) \geq 0$, and pseudoinvexity yields $f_i(x) \geq f_i(u)$. Thus no feasible x strictly improves every objective component, which proves weak Pareto optimality.

Following the multiobjective bundle construction in [16], define the improvement function

$$H(x, y) = \max \left\{ f_i(x) - f_i(y), g_j(x) : i = 1, \dots, p, j = 1, \dots, m \right\}. \quad (15)$$

Theorem 3. *Suppose that the objective functions f_i are pseudoinvex, the constraint functions g_j are quasinvex with respect to the same η , and (13) holds at x^* . If*

$$x^* \in \arg \min_{x \in \mathbb{R}^n} H(x, x^*), \quad (16)$$

then x^* is a global weakly Pareto optimal solution of (7).

Proof. Since x^* minimizes $H(\cdot, x^*)$, Clarke stationarity gives $0 \in \partial_x H(x^*, x^*)$. The subdifferential of the finite maximum is contained in the convex hull of the active objective and constraint subdifferentials, so

$$0 \in \text{conv} \{ \mathcal{F}(x^*) \cup \mathcal{G}(x^*) \}.$$

The constraint qualification excludes a representation using only active constraint subgradients. Consequently the objective part has a positive total coefficient; after normalization we obtain $0 \in \text{conv } \mathcal{F}(x^*) + \text{cone } \mathcal{G}(x^*)$. The conclusion follows from Theorem 2.

5 Multiobjective proximal bundle method

Let x^h be the current serious iterate at iteration h . The ideal search direction would solve

$$\min_{d \in \mathbb{R}^n} H(x^h + d, x^h), \quad (17)$$

which is generally nonsmooth and nonconvex. A bundle model replaces each component by cutting planes built at auxiliary points y^j retained from previous iterations.

For $j \in J_h \subset \{1, \dots, h\}$, choose

$$\xi_{f_i}^j \in \partial f_i(y^j), \quad \xi_{g_l}^j \in \partial g_l(y^j).$$

The affine localizations are

$$f_{i,j}(x) = f_i(y^j) + \langle \xi_{f_i}^j, x - y^j \rangle, \quad (18)$$

$$g_{l,j}(x) = g_l(y^j) + \langle \xi_{g_l}^j, x - y^j \rangle. \quad (19)$$

At x^h , define linearization errors

$$a_{f_i,j}^h = f_i(x^h) - f_{i,j}(x^h), \quad (20)$$

$$a_{g_l,j}^h = -g_{l,j}(x^h). \quad (21)$$

For nonconvex functions we replace these by the locality measures

$$\beta_{f_i,j}^h = \max\{|a_{f_i,j}^h|, \gamma_{f_i} \|x^h - y^j\|^2\}, \quad (22)$$

$$\beta_{g_l,j}^h = \max\{|a_{g_l,j}^h|, \gamma_{g_l} \|x^h - y^j\|^2\}, \quad (23)$$

where $\gamma_{f_i}, \gamma_{g_l} \geq 0$; one may set the corresponding γ to zero for convex components.

The convex piecewise-linear approximation of the improvement function leads to the proximal subproblem

$$\min_{d \in \mathbb{R}^n} \hat{H}_h(x^h + d) + \frac{u_h}{2} \|d\|^2, \quad u_h > 0. \quad (24)$$

Equivalently, introducing an epigraph variable v , one solves the strictly convex quadratic program

$$\begin{aligned} \min_{d,v} \quad & v + \frac{u_h}{2} \|d\|^2, \\ \text{s.t.} \quad & -\beta_{f_i,j}^h + \langle \xi_{f_i}^j, d \rangle \leq v, \quad i = 1, \dots, p, \quad j \in J_h, \\ & -\beta_{g_l,j}^h + \langle \xi_{g_l}^j, d \rangle \leq v, \quad l = 1, \dots, m, \quad j \in J_h. \end{aligned} \quad (25)$$

The proximal term both makes the subproblem strongly convex and keeps the model local. Denote its unique solution by (d^h, v^h) .

5.1 Two-point line search

Choose line-search parameters

$$0 < m_L < \frac{1}{2}, \quad m_L < m_R < 1, \quad \bar{t} \in (0, 1].$$

A serious-step length $t_L^h \in [0, 1]$ is selected so that

$$\max_i \{f_i(x^h + t_L^h d^h) - f_i(x^h)\} \leq m_L t_L^h v^h, \quad (26)$$

$$\max_l g_l(x^h + t_L^h d^h) \leq 0. \quad (27)$$

The method distinguishes three cases. If $t_L^h \geq \bar{t}$, a long serious step is accepted and the trial point equals the new serious point. If $0 < t_L^h < \bar{t}$, a short serious step updates the serious point but a farther trial point $y^{h+1} = x^h + t_R^h d^h$, $t_R^h > t_L^h$, is used to enrich the bundle. If $t_L^h = 0$, a null step keeps $x^{h+1} = x^h$ while a new trial point is generated. The right-hand step is chosen so that new subgradient information gives sufficient model improvement; a representative condition is

$$-\beta_{f_i, h+1}^{h+1} + \langle \xi_{f_i}^{h+1}, d^h \rangle \geq m_R v^h \quad (28)$$

for an appropriate objective component. This two-point strategy follows the nonconvex bundle philosophy in [14, 16].

5.2 Algorithm

Algorithm 1 (MPBM).

Step 1. Initialization. Choose a feasible x^1 , tolerance $\varepsilon_s > 0$, proximal weight $u_1 > 0$, line-search parameters $m_L, m_R, \bar{t}$, and locality parameters $\gamma_{f_i}, \gamma_{g_l} \geq 0$. Set $h = 1$, $y^1 = x^1$, and compute one Clarke subgradient of every objective and constraint component at y^1 .

Step 2. Direction finding. Form the current bundle and solve (25) to obtain (d^h, v^h) .

Step 3. Stopping test. If

$$-\frac{1}{2}v^h < \varepsilon_s, \quad (29)$$

stop.

Step 4. Line search. Determine t_L^h and t_R^h according to (26)–(28), set

$$x^{h+1} = x^h + t_L^h d^h, \quad y^{h+1} = x^h + t_R^h d^h.$$

Step 5. Update. Compute subgradients at y^{h+1} , update the bundle index set J_{h+1} and the proximal weight u_{h+1} , increment h , and return to Step 2.

To keep the bundle size bounded, one may use subgradient aggregation as in [11]. Proximity-control updates are discussed in [13]; practical nonsmooth implementations and nonconvex modifications are described in [14, 2].

6 Convergence under generalized invexity

The crucial observation is that the MPBM applied to (7) can be viewed as a proximal bundle method for the scalar improvement function $H(\cdot, x^h)$, with the vector structure encoded by the maximum. This makes it possible to combine standard bundle convergence with global consequences of pseudoinvexity.

Theorem 4 (Global weak Pareto convergence). *Assume that every objective function f_i and every constraint function g_j is locally Lipschitz, pseudoinvex and weakly semismooth, with the pseudoinvexity taken with respect to a common mapping η . Assume also that the constraint qualification (13) is valid at the relevant limit points. Then:*

- (i) *if Algorithm 1 terminates after finitely many iterations at x^h with exact stationarity in the zero-tolerance limit, the resulting point is a global weakly Pareto optimal solution of (7);*
- (ii) *if Algorithm 1 generates an infinite sequence and $\bar{x}$ is an accumulation point satisfying the bundle stationarity limit, then $\bar{x}$ is a global weakly Pareto optimal solution of (7).*

Proof. By Theorem 1, a finite maximum of the pseudoinvex functions defining (15) is pseudoinvex with respect to the common η . Thus, for fixed second argument, the improvement function inherits pseudoinvexity. The convergence theory of proximal bundle methods for weakly semismooth nonsmooth functions implies that finite exact termination, or the corresponding stationarity limit along a convergent subsequence, gives

$$0 \in \partial_x H(\bar{x}, \bar{x});$$

see [13, 28, 16]. Pseudoinvexity converts this stationarity condition into global minimality of $H(\cdot, \bar{x})$. Theorem 3, together with the constraint qualification, then yields global weak Pareto optimality of $\bar{x}$.

Remark 2. For a finite positive tolerance ε_s , Algorithm 1 returns an approximate bundle-stationary point rather than an exact stationary point. Theorem 4 should therefore be understood as an exact convergence statement, while (29) supplies a numerical stopping criterion. This distinction is important when reporting computational solutions.

7 Specialization to nonsmooth fuzzy optimization

Consider the fuzzy-valued constrained problem

$$\begin{aligned} & \text{minimize} && \tilde{f}(x), \\ & \text{subject to} && g_j(x) \leq 0, \quad j = 1, \dots, m, \end{aligned} \tag{30}$$

where the α -cut of the objective is given by (10). For any fixed α , define

$$f_1^\alpha(x) = f^L(x, \alpha), \quad f_2^\alpha(x) = f^R(x, \alpha).$$

Then (30) produces the two-objective problem

$$\text{minimize } (f_1^\alpha(x), f_2^\alpha(x)) \quad \text{subject to } g_j(x) \leq 0. \quad (31)$$

If both endpoint functions and the constraints satisfy the assumptions of Theorem 4, Algorithm 1 can be applied directly with $p = 2$. The bundle now contains subgradients of the two endpoint functions together with those of the constraints.

This level-wise viewpoint is attractive for three reasons. First, all optimization calculations remain in real finite-dimensional spaces. Second, no differentiability of the fuzzy endpoints is required; Clarke subgradients are sufficient. Third, generalized invexity promotes the stationary solutions produced asymptotically by the bundle procedure to global weak Pareto solutions of (31). Repeating the computation over a set of α -levels reveals whether the same decision remains nondominated as the confidence/possibility level varies.

8 Computational illustration

We illustrate the approach using a nonsmooth fuzzy problem with a triangular fuzzy parameter. Let $x = (x_1, x_2) \in \mathbb{R}^2$ and consider

$$\text{minimize } \tilde{f}(x) = \sqrt{\|x\|} + \tilde{2}, \quad (32)$$

subject to

$$g(x) = \max\{x_1^2 + x_2^2 - 10, 3x_1 + x_2 + 1.5\} \leq 0, \quad (33)$$

where $\tilde{2} = (0, 2, 4)$ is triangular. From (9), $[\tilde{2}]_\alpha = [2\alpha, 4(1 - \alpha) + 2\alpha]$, so the two endpoint objectives are

$$f^L(x, \alpha) = \sqrt{\|x\|} + 2\alpha, \quad f^R(x, \alpha) = \sqrt{\|x\|} + 4(1 - \alpha) + 2\alpha. \quad (34)$$

The starting point is $x^0 = (-1, 0)$. Table 1 reports selected iterates for two representative levels. The full run converges toward approximately the same decision for $\alpha = 0.1$ and $\alpha = 0.9$.

Table 1. Selected MPBM iterates for the fuzzy problem (32)–(33).

α	iteration	x_1	x_2	(f^L, f^R)
0.1	0	-1.0000000	0.0000000	(1.0954450, 2.190890)
0.1	2	-0.4552567	-0.1433555	(0.8229787, 2.068162)
0.1	6	-0.4542455	-0.1374579	(0.8213330, 2.067508)
0.1	10	-0.4512369	-0.1463066	(0.8211961, 2.067453)
0.9	0	-1.0000000	0.0000000	(1.673320, 1.788854)
0.9	2	-0.4501177	-0.1589698	(1.509094, 1.636266)
0.9	8	-0.4479432	-0.1562289	(1.508113, 1.635361)
0.9	14	-0.4491176	-0.1526574	(1.508096, 1.635345)

For $\alpha = 0.1$, the tenth iterate is approximately $(-0.45124, -0.14631)$; for $\alpha = 0.9$, the fourteenth iterate is approximately $(-0.44912, -0.15266)$. Both runs therefore approach approximately $(-0.45, -0.15)$. This agreement is useful from the fuzzy-optimization perspective because it suggests robustness of the candidate with respect to the selected α -level. A rigorous F-nondominance statement over the continuum $[0, 1]$, however, requires the corresponding endpoint dominance condition for all α , not only for the two levels displayed here.

9 Discussion and conclusion

The proposed framework connects four ingredients that are often treated separately: fuzzy α -cut modelling, nonsmooth Clarke calculus, generalized invexity and multiobjective bundle optimization. The α -cut transformation is computationally convenient because it turns a fuzzy-valued objective into two ordinary nonsmooth objectives. The max-type improvement function then brings the multiobjective problem back to a scalar bundle model while preserving enough vector information to derive weak Pareto conclusions.

Generalized invexity changes the interpretation of bundle stationarity in an essential way. For a generic nonconvex nonsmooth problem, a stationary limit may only be locally meaningful. Under the common- η pseudoinvexity assumptions used here, stationarity of the improvement function implies global minimality and hence global weak Pareto optimality. This is the main theoretical advantage of combining the invex structure with MPBM. At the same time, the common- η assumption is substantive and should be verified rather than treated as automatic.

Several extensions appear natural. Adaptive selection of α -levels could reduce the cost of checking level-wise fuzzy nondominance. Alternative scalarizations could be incorporated into the bundle model, especially when a decision maker wishes to explore different parts of the Pareto set. One may also investigate other generalized differentiation frameworks, including limiting/Mordukhovich subdifferentials or quasidifferential calculus, although the max and sum rules required by the present proofs would then need to be reconsidered carefully. Finally, large-scale implementations would benefit from bundle compression, specialized quadratic-programming solvers and parallel evaluation of endpoint functions and constraints.

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