

Post-identification Lyapunov Certification of Sparse Quadratic Models^{*}

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Abstract. This paper presents a post-identification certification framework for sparse quadratic dynamical models identified from data. After model selection, the coefficients, active terms, data partitions, and preprocessing choices are frozen so that certification does not alter the identified dynamics. The frozen model is examined using a Hurwitz eigenvalue screen, a fixed Euclidean Lyapunov metric, and a post-identification quadratic Lyapunov metric obtained through semidefinite optimization. The quadratic nonlinear terms are then included in the Lyapunov derivative analysis of the full model, yielding a conservative local region in which the Lyapunov function is guaranteed to decrease. Coefficient uncertainty is treated separately through robust certification over calibrated uncertainty sets. The framework therefore separates predictive validation, nominal stability certification, local nonlinear certification, and robust certification, providing a transparent way to assess an already identified model without modifying it.

Keywords: post-identification certification · SINDy · Lyapunov stability · sparse nonlinear models · uncertainty quantification · semidefinite programming.

1 Introduction

Data-driven identification provides a practical way to construct dynamical models when the governing equations or their parameters are unknown or only partially known. Among these methods, Sparse Identification of Nonlinear Dynamics (SINDy) constructs compact equations by selecting a small number of active terms from a larger set of candidate functions [4, 3]. SINDy can use many types of candidate functions. Here, we restrict the library to constant, linear, and quadratic polynomial terms because the certification framework is developed for sparse quadratic models. This restriction defines the model class studied below.

Accurately reproducing measured data, however, does not by itself show that the identified dynamics have desirable stability properties. Measurement noise

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and numerical differentiation can affect the estimated derivatives, and the selected equation structure also depends on the candidate functions used during identification [5, 15]. An identified model may accurately reproduce measured derivatives or short trajectories yet generalize poorly beyond the observed data. For this reason, prediction accuracy and dynamical stability should be examined as separate questions.

Several existing approaches introduce stability or boundedness conditions while the model is being identified. For example, Trapping SINDy includes conditions intended to keep trajectories inside a bounded region for a class of quadratic systems [11]. Other methods impose stability-related constraints directly during the construction of quadratic models [9, 7]. Related identification work has also embedded constraints directly into the model structure [16].

Once a model has been selected, a separate question is whether its stability can be certified without altering the identified dynamics. Lyapunov-based methods address this question and have been applied to fixed quadratic models to estimate stability regions, bound nonlinear terms, and study sparse models under noisy-data settings [13, 17, 19]. The proposed workflow applies complementary tests to the frozen model. First, we screen the linearized dynamics for the Hurwitz property. Second, we test a fixed Euclidean Lyapunov metric. Third, we compute a post-identification quadratic Lyapunov metric for the frozen dynamics. Combined with the quadratic nonlinear bound introduced below, the resulting inequality yields a conservative local region of guaranteed Lyapunov decrease.

To account for uncertainty in coefficients estimated from noisy data, separate calibration data are used to define bounds on coefficient errors. Robust post-identification certification then tests whether one Lyapunov metric remains valid over the resulting uncertainty set. The failure of this sufficient test indicates that the tested certificate cannot be guaranteed over the prescribed uncertainty range.

The main contribution is a transparent post-identification certification workflow that freezes the selected sparse quadratic model and separates predictive validation, nominal stability certification, and robust certification. Its purpose is to determine which stability-related claims are supported for the identified model and over what region. Passing the Hurwitz screen confirms stability of the linearized dynamics; passing a nominal Lyapunov test together with the nonlinear bound yields a local region of guaranteed Lyapunov decrease; and passing the robust test extends the certificate to all models in the prescribed uncertainty set.

2 Sparse Quadratic Model and Frozen Setting

We consider a continuous-time autonomous dynamical system,

$$\dot{x}(t) = f(x(t)), \quad x(t) \in \mathbb{R}^n, \quad (1)$$

where $x(t)$ is the n -dimensional state vector and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the unknown vector field.

The continuous-time state trajectory is observed at m discrete time instants $t_1, \dots, t_m$. The measured states and their derivatives are arranged into the matrices

$$X = \begin{bmatrix} x(t_1)^\top \\ \vdots \\ x(t_m)^\top \end{bmatrix}, \quad \dot{X} = \begin{bmatrix} \dot{x}(t_1)^\top \\ \vdots \\ \dot{x}(t_m)^\top \end{bmatrix}, \quad X, \dot{X} \in \mathbb{R}^{m \times n}. \quad (2)$$

When derivatives are not measured directly, $\dot{X}$ can be estimated from the measured states using numerical differentiation.

Sparse Identification of Nonlinear Dynamics (SINDy) represents the unknown vector field $f(x(t))$ using a library of candidate functions [4]. Let $\theta(x) \in \mathbb{R}^{1 \times p}$ be the row vector containing the p candidate functions evaluated at state x . Evaluating these functions at all m samples gives

$$\Theta(X) = \begin{bmatrix} \theta(x(t_1)) \\ \vdots \\ \theta(x(t_m)) \end{bmatrix} \in \mathbb{R}^{m \times p}. \quad (3)$$

The identified dynamics satisfy

$$\dot{X} \approx \Theta(X)\Xi, \quad \Xi \in \mathbb{R}^{p \times n}, \quad (4)$$

where the j -th column of Ξ contains the coefficients of the j -th state equation.

Assuming that only a few candidate functions contribute to the dynamics, sparse regression seeks a coefficient matrix with few nonzero entries, whose locations define the active support of the identified model. A general formulation is

$$\min_{\Xi} \frac{1}{2m} \left\| \dot{X} - \Theta(X)\Xi \right\|_F^2 + \lambda \mathcal{R}(\Xi), \quad (5)$$

where $\|\cdot\|_F$ is the Frobenius norm, $\mathcal{R} : \mathbb{R}^{p \times n} \rightarrow \mathbb{R}_{\geq 0}$ is a sparsity-promoting regularizer, and $\lambda \geq 0$ sets its weight.

In this paper, the candidate library contains a constant term, the n individual state variables, and all distinct quadratic products $x_j x_k$ with $1 \leq j \leq k \leq n$. Thus, the number of candidate functions is

$$p = 1 + n + \frac{n(n+1)}{2}. \quad (6)$$

After model selection, the identified dynamics can be written as

$$\dot{x} = c + Ax + h(x), \quad h_i(x) = x^\top H_i x, \quad i = 1, \dots, n, \quad (7)$$

where $c \in \mathbb{R}^n$ contains the constant coefficients, $A \in \mathbb{R}^{n \times n}$ contains the linear coefficients, $h(x) \in \mathbb{R}^n$ collects the quadratic contributions, and $H_i \in \mathbb{R}^{n \times n}$ is the coefficient matrix of the quadratic terms in the i -th state equation. Without loss of generality, each H_i is chosen symmetric, since $x^\top H_i x$ depends only on

its symmetric part. This gives a unique matrix representation of each quadratic polynomial,

$$H_i = H_i^\top. \quad (8)$$

For a cross term $\beta_{jk}^{(i)} x_j x_k$, where $\beta_{jk}^{(i)} \in \mathbb{R}$ is its coefficient and $1 \leq j < k \leq n$, we set

$$(H_i)_{jk} = (H_i)_{kj} = \frac{\beta_{jk}^{(i)}}{2}. \quad (9)$$

The diagonal entries of H_i contain the coefficients of the squared terms.

Certification is performed around an equilibrium $x_e \in \mathbb{R}^n$ of the identified model, satisfying

$$0 = c + Ax_e + h(x_e). \quad (10)$$

We introduce the shifted state

$$\tilde{x} = x - x_e, \quad \tilde{x} \in \mathbb{R}^n, \quad (11)$$

which moves the equilibrium to the origin. Under this coordinate shift, the quadratic terms generate additional terms linear in $\tilde{x}$, so the linear matrix of the shifted dynamics generally differs from A .

Let $a_i \in \mathbb{R}^n$ denote the transpose of the i -th row of A . Substituting $x = \tilde{x} + x_e$ into Eq. (7) gives

$$\dot{\tilde{x}} = L\tilde{x} + h(\tilde{x}), \quad L \in \mathbb{R}^{n \times n}, \quad (12)$$

where L is the linear part of the identified model around the equilibrium x_e . Its i -th row, $L_{i,:} \in \mathbb{R}^{1 \times n}$, is

$$L_{i,:} = (a_i + 2H_i x_e)^\top. \quad (13)$$

Expanding $x^\top H_i x$ after the shift produces the linear cross term $2x_e^\top H_i \tilde{x}$, which contributes $2H_i x_e$ to the i -th linear coefficient vector. Using the equilibrium condition, the constant terms cancel, the linear matrix changes from A to L , and the quadratic matrices H_i remain unchanged.

After model selection and the equilibrium shift, L , H_i , the active support, fitted coefficients, data partitions, and preprocessing choices are frozen. The frozen model in Eq. (12) is then evaluated using the stability tests introduced in the next section.

3 Lyapunov Certification of the Frozen Model

We now examine the frozen shifted model in Eq. (12). Throughout this section, $\|\cdot\|_2$ denotes the Euclidean norm for vectors and the spectral norm for matrices. For symmetric matrices, $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denote the largest and smallest eigenvalues, and I denotes an identity matrix of appropriate dimension.

A Lyapunov certificate consists of a positive-definite metric P and inequalities guaranteeing Lyapunov decrease for the frozen model over a specified region. The Hurwitz eigenvalue check is used separately to screen the linearization.

3.1 Hurwitz Screen and Fixed Euclidean Metric

We first examine the eigenvalues of L in Eq. (12). The spectral abscissa is defined as

$$\alpha(L) = \max_{j=1,\dots,n} \operatorname{Re}(\lambda_j(L)), \quad (14)$$

where $\lambda_j(L) \in \mathbb{C}$ is the j -th eigenvalue of L and $\operatorname{Re}(\cdot)$ denotes its real part. The matrix L is Hurwitz when

$$\alpha(L) < 0. \quad (15)$$

The Hurwitz test screens the linearization independently of any Lyapunov metric. We then consider the Euclidean quadratic candidate

$$V_I(\tilde{x}) = \frac{1}{2} \tilde{x}^\top \tilde{x}. \quad (16)$$

This corresponds to the fixed Euclidean metric $P = I$. For the linearized dynamics

$$\dot{\tilde{x}} = L\tilde{x},$$

the derivative is

$$\dot{V}_I(\tilde{x}) = \frac{1}{2} \tilde{x}^\top (L^\top + L) \tilde{x}. \quad (17)$$

Thus, the fixed metric provides a strict Lyapunov certificate for the linearized dynamics when

$$L^\top + L \prec 0. \quad (18)$$

Equivalently,

$$\tilde{x}^\top (L^\top + L) \tilde{x} < 0$$

for every nonzero $\tilde{x} \in \mathbb{R}^n$ [12, 2]. For numerical comparison, we define the fixed Euclidean margin

$$\eta_I = -\lambda_{\max}(L^\top + L). \quad (19)$$

Since $L^\top + L$ is symmetric, if $\eta_I > 0$,

$$\dot{V}_I(\tilde{x}) \leq -\frac{\eta_I}{2} \|\tilde{x}\|_2^2 \quad (20)$$

for every nonzero $\tilde{x}$ in the linearized dynamics. This is the standard continuous-time quadratic Lyapunov condition for the fixed Euclidean metric [12, 2].

If $\eta_I \leq 0$, the fixed Euclidean metric does not provide a certificate. This does not imply that L is unstable; it only means that $P = I$ is insufficient to establish strict Lyapunov decay.

3.2 Post-Identification Quadratic Metric

We next use the quadratic Lyapunov candidate

$$V_P(\tilde{x}) = \frac{1}{2} \tilde{x}^\top P \tilde{x}, \quad P = P^\top \succ 0, \quad P \in \mathbb{R}^{n \times n}. \quad (21)$$

The matrix P defines the Lyapunov geometry.¹

For the linearized dynamics,

$$\dot{V}_P(\tilde{x}) = \frac{1}{2} \tilde{x}^\top (L^\top P + PL) \tilde{x}. \quad (22)$$

Given $P = P^\top \succ 0$, a strict quadratic Lyapunov certificate for the linearized dynamics is obtained when

$$L^\top P + PL \prec 0. \quad (23)$$

[12, 2]

Since the scale of P is arbitrary, we impose

$$\text{tr}(P) = n. \quad (24)$$

The post-identification metric is obtained by solving the semidefinite program (SDP) [2]

$$\begin{aligned} \max_{P, \eta} \quad & \eta \\ \text{subject to} \quad & P = P^\top, \quad P \succeq \delta I, \\ & \text{tr}(P) = n, \\ & L^\top P + PL \preceq -\eta I. \end{aligned} \quad (25)$$

This is a convex optimization problem because the objective is linear and the matrix inequalities are affine in the decision variables P and η . Here, $\eta \in \mathbb{R}$ is the linear decay margin optimized by the SDP, and $\delta > 0$ is a fixed numerical safeguard that keeps P uniformly positive definite.

Let $P^\star$ denote an optimal metric obtained from the SDP and $\eta^\star$ the corresponding optimal decay margin. If $\eta^\star > 0$, then

$$\dot{V}_{P^\star}(\tilde{x}) \leq -\frac{\eta^\star}{2} \|\tilde{x}\|_2^2 \quad (26)$$

for every nonzero $\tilde{x}$ along the frozen linearized dynamics.

3.3 Local Nonlinear Certificate

The previous tests concern the linear term $L\tilde{x}$ of the frozen shifted model in Eq. (12). We now include the quadratic nonlinear terms.

$$h_i(\tilde{x}) = \tilde{x}^\top H_i \tilde{x}, \quad H_i = H_i^\top \in \mathbb{R}^{n \times n}. \quad (27)$$

¹ For $P = I$, the level sets are spheres; for general $P \succ 0$, they are ellipsoids.

To characterize their combined magnitude, define

$$\bar{\gamma}_H = \left(\sum_{i=1}^n \|H_i\|_2^2 \right)^{1/2}. \quad (28)$$

It follows that [13, 17]

$$\|h(\tilde{x})\|_2 \leq \bar{\gamma}_H \|\tilde{x}\|_2^2. \quad (29)$$

Let $P = P^\top \succ 0$ satisfy the linear Lyapunov inequality with decay margin $\eta > 0$,

$$L^\top P + PL \preceq -\eta I. \quad (30)$$

For $V_P(\tilde{x}) = \frac{1}{2} \tilde{x}^\top P \tilde{x}$, the derivative along the full frozen nonlinear model in Eq. (12) satisfies

$$\begin{aligned} \dot{V}_P(\tilde{x}) &= \frac{1}{2} \tilde{x}^\top (L^\top P + PL) \tilde{x} + \tilde{x}^\top P h(\tilde{x}) \\ &\leq -\frac{\eta}{2} \|\tilde{x}\|_2^2 + \|P\|_2 \bar{\gamma}_H \|\tilde{x}\|_2^3. \end{aligned} \quad (31)$$

Case 1: $\bar{\gamma}_H > 0$. The upper bound is strictly negative for every nonzero $\tilde{x}$ satisfying

$$\|\tilde{x}\|_2 < r_{\dot{V}}(P), \quad r_{\dot{V}}(P) = \frac{\eta}{2\|P\|_2 \bar{\gamma}_H}. \quad (32)$$

We call $r_{\dot{V}}(P)$ the derivative-negativity radius, which guarantees $\dot{V}_P(\tilde{x}) < 0$ for every nonzero $\tilde{x}$ with $\|\tilde{x}\|_2 < r_{\dot{V}}(P)$. It does not by itself define an invariant set or the exact region of attraction.

For $\bar{\gamma}_H > 0$, choose a safety factor $\rho \in (0, 1)$ and define

$$c_\rho = \lambda_{\min}(P) (\rho r_{\dot{V}}(P))^2. \quad (33)$$

The corresponding Lyapunov sublevel set is

$$\mathcal{E}_\rho(P) = \{ \tilde{x} \in \mathbb{R}^n : \tilde{x}^\top P \tilde{x} \leq c_\rho \}. \quad (34)$$

Since $\tilde{x}^\top P \tilde{x} \geq \lambda_{\min}(P) \|\tilde{x}\|_2^2$, every $\tilde{x} \in \mathcal{E}_\rho(P)$ satisfies

$$\|\tilde{x}\|_2 \leq \rho r_{\dot{V}}(P) < r_{\dot{V}}(P).$$

Hence, $\dot{V}_P(\tilde{x}) < 0$ at every nonzero point of $\mathcal{E}_\rho(P)$. Because this set is a sublevel set of V_P , it is positively invariant for the frozen nominal model [12].

Case 2: $\bar{\gamma}_H = 0$. All quadratic terms vanish, and the frozen model is linear. In this case, the linear Lyapunov certificate applies globally, and no finite nonlinear radius is required. The fixed and post-identification Lyapunov margins, nonlinear radii, and ellipsoid dimensions depend on the coordinates and scaling of the state variables. They are therefore compared only for models expressed in the same coordinates and scaling.

The nominal certificates derived above apply to the frozen model; coefficient uncertainty is addressed separately in Section 4.

4 Numerical Results

The numerical study addresses three questions. First, we examine the SINDy identification-to-certification workflow under ideal conditions and the effect of sparsity. Second, we test whether the fixed Euclidean metric can fail to provide a certificate for Hurwitz but non-normal linear dynamics. Third, we examine how coefficient uncertainty affects nominal and robust certification. In each case, the fitted model is kept fixed during certification.

4.1 Baseline and Sparsity Checks

The baseline uses the two-dimensional continuous-time quadratic autonomous system

$$\begin{aligned}\dot{x}_1 &= -x_1 + 0.5x_1x_2, \\ \dot{x}_2 &= -2x_2 - 0.3x_1^2.\end{aligned}\tag{35}$$

Four noise-free trajectories are generated on $t \in [0, 10]$ with $\Delta t = 0.01$ from initial conditions $(1.0, 0.5)$, $(-0.8, 0.7)$, $(0.6, -0.6)$, and $(-0.5, -0.4)$. The first three are used for training and trajectory-wise cross-validation, and the fourth for held-out evaluation. Exact derivatives are evaluated directly from the right-hand side of Eq. (35), without numerical differentiation. The quadratic SINDy library is $[1, x_1, x_2, x_1^2, x_1x_2, x_2^2]$.

Sequentially thresholded least squares (STLSQ) is evaluated on the following a fixed grid of 15 threshold values:

$$A = \{10^{-8}, 10^{-6}, 10^{-4}, 10^{-3}, 10^{-2}, 0.05, 0.1, 0.2, 0.25, 0.35, 0.45, 0.55, 0.9, 1.1, 1.5\}.$$

Three leave-one-trajectory-out cross-validation folds are used: in each fold, two trajectories are used for fitting and the third for validation. The derivative RMSE is computed over all validation samples and state components and then averaged over the three folds. The one-standard-error rule selects the sparsest admissible model [4, 10]. The selected threshold is $\lambda_{\text{sel}} = 0.25$, yielding four active coefficients and recovering the reference support and nonzero coefficients to numerical precision.

Performance is then evaluated on the fourth trajectory, which was not used for training or model selection. The held-out derivative error measures how accurately the frozen model predicts the state derivatives along this unseen trajectory, while the held-out state error measures the difference between the forward-simulated and reference trajectories.

$$\text{RMSE}_{\dot{x}} = 2.10 \times 10^{-13}, \quad \text{RMSE}_x = 1.08 \times 10^{-13}.$$

These near-numerical-precision errors are expected for this noise-free exact-recovery benchmark and mainly confirm the implementation of the identification-to-certification pipeline. The identified linearization has $\alpha(L) = -1 < 0$. The fixed and post-identification Lyapunov margins are $\eta_I = 2.00$ and $\eta^* = 2.67$,

respectively, so both metrics certify the frozen linearized dynamics. The corresponding local nonlinear certificates are then obtained by including the quadratic terms. Although the adaptive margin is larger, the corresponding metric norm also increases, and both metrics give approximately the same derivative-negativity radius, $r_{\dot{V}} \approx 2.56$.

Across the full threshold sweep, nine of the 15 candidates satisfy the predictive one-standard-error rule. Thirteen candidates pass both Lyapunov tests, while two pass neither. All nine predictively admissible candidates pass the certification tests. Thus, in this ideal baseline, post-identification certification does not further discriminate among the predictively admissible models.

4.2 Effect of Non-Normal Coupling

We next illustrate how non-normal coupling affects the fixed Euclidean certificate and the resulting local nonlinear bound while the eigenvalues remain unchanged.

$$L(a) = \begin{bmatrix} -1 & a \\ 0 & -2 \end{bmatrix}, \quad a \geq 0, \quad (36)$$

where the scalar a controls the off-diagonal linear coupling. The quadratic terms remain those of Eq. (35). We test $a \in \{0, 1, 2, 2.5, 2.8, 3, 3.5, 4\}$. The case $a = 0$ provides the normal baseline, while $a > 0$ introduces non-normal coupling without changing the eigenvalues. For each value of a , six noise-free trajectories are generated on $t \in [0, 6]$ with $\Delta t = 0.01$, and exact derivatives are evaluated from the corresponding right-hand side. SINDy uses the same quadratic library as in Section 4.1, but with the STLSQ threshold fixed at $\lambda = 0.25$, rather than selected again by cross-validation. The corresponding sparse quadratic model is recovered to numerical precision in every case.

Because $L(a)$ is upper triangular, its eigenvalues remain -1 and -2 for every a . Thus, every tested linearization is Hurwitz, although increasing a changes the geometry of the linear dynamics [20]. For the fixed Euclidean metric,

$$\eta_I(a) = 3 - \sqrt{1 + a^2}. \quad (37)$$

The fixed Euclidean condition holds through $a = 2.8$ and fails for $a = 3, 3.5, 4$. The post-identification SDP yields $\eta^* > 0$ for all eight tested values. Thus, all tested linearizations remain Hurwitz, while the fixed Euclidean metric becomes insufficient as the non-normal coupling increases. The post-identification SDP continues to provide a positive Lyapunov margin for the same frozen dynamics.

The SDP is specified in CVXPY [6] and solved with CLARABEL [8] using $\delta = 10^{-6}$. A solution is accepted only if $\eta^* > 10^{-6}$, the trace and normalized LMI-residual errors are at most 10^{-5} , and $\lambda_{\min}(P^*) \geq \delta - 10^{-8}$.

The same distinction carries over to the local nonlinear certificate. At $a = 2.8$, the fixed and post-identification radii are approximately 0.0343 and 1.0581, respectively. For $a = 3, 3.5, 4$, the fixed Euclidean certificate fails, so no corresponding local radius is reported, whereas the post-identification radii remain approximately 0.9781, 0.8080, and 0.6736, respectively.

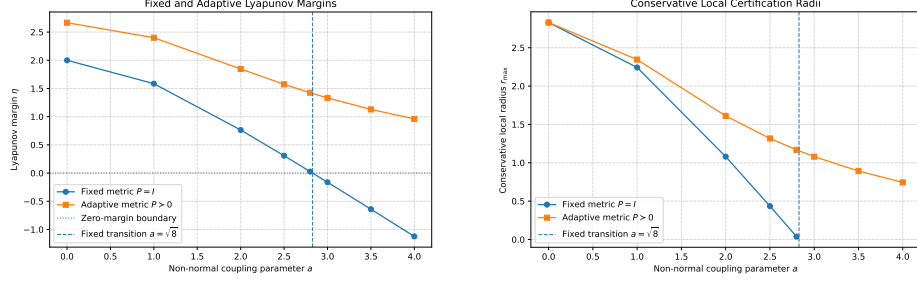


Fig. 1. Effect of non-normal coupling on the fixed and post-identification Lyapunov margins (left) and derivative-negativity radii (right). The eigenvalues remain -1 and -2 for all a , while the fixed Euclidean certificate is lost at $a = \sqrt{8}$.

4.3 Coefficient Uncertainty and Robust Certification

The final study uses a ten-dimensional synthetic quadratic benchmark with equilibrium at the origin.² Eight clean trajectories are generated on $t \in [0, 8]$ with $\Delta t = 0.01$; the first six are used for clean identification and the remaining two are held out. A 47-coefficient support is identified from the clean data and then frozen. In the noisy experiments, no terms are added or removed; only the coefficients on this fixed support are re-estimated.

Two noise levels are considered, $\sigma \in \{0.005, 0.020\}$. For each state component j , noisy measurements are generated as

$$x_j^{\text{noisy}}(t_k) = x_j(t_k) + \sigma s_j z_{j,k}, \quad z_{j,k} \sim \mathcal{N}(0, 1),$$

where s_j is the standard deviation of the j -th state component computed from the six clean training trajectories.

Noise is added only to the sampled states; derivatives are not perturbed directly but are estimated from the noisy states using a cubic smoothing spline with generalized cross-validation. For evaluation, 100 realizations disjoint from the calibration set are reserved at each noise level. For a given seed, the same standardized noise realization is rescaled to the two noise levels. Since the benchmark is synthetic, the reference operators are known exactly. This allows the errors in the estimated linear and quadratic operators to be measured as

$$\begin{aligned} e_L &= \|\hat{L} - L_{\text{true}}\|_2, \\ e_H &= \left(\sum_{i=1}^{10} \|\hat{H}_i - H_{\text{true},i}\|_2^2 \right)^{1/2}. \end{aligned} \quad (38)$$

² The complete deterministic benchmark specification, including the reference linear and quadratic operators and all initial conditions, can be made available by the authors upon reasonable request.

Here, $\hat{L}, L_{\text{true}} \in \mathbb{R}^{10 \times 10}$ are the estimated and reference linear operators. Similarly, $\hat{H}_i, H_{\text{true},i} \in \mathbb{R}^{10 \times 10}$, $i = 1, \dots, 10$, are the estimated and reference quadratic operators.

Separately, 40 calibration realizations at each noise level are used to construct a joint split-conformal bound [14]. Let $e_L^{(r)}$ and $e_H^{(r)}$ denote the linear and quadratic operator errors of the r -th calibration realization. For each noise level, a_L and a_H are fixed before calibration as the medians of the corresponding linear and quadratic operator errors over four pilot realizations, with a small deterministic positive floor. We then define,

$$s_r = \max \left\{ \frac{e_L^{(r)}}{a_L}, \frac{e_H^{(r)}}{a_H} \right\}, \quad r = 1, \dots, 40.$$

The joint conformal quantile q is the 39th smallest of these 40 scores, corresponding to the prescribed 95% coverage level. The uncertainty radii are then frozen as

$$\varepsilon_L = q a_L, \quad \varepsilon_H = q a_H.$$

At $\sigma = 0.005$, $(\varepsilon_L, \varepsilon_H) = (7.423 \times 10^{-2}, 1.095)$; at $\sigma = 0.020$, they increase to $(1.941 \times 10^{-1}, 2.596)$. These radii are fixed before the evaluation realizations are examined.

For an estimated model $(\hat{L}, \hat{H}_1, \dots, \hat{H}_{10})$, the corresponding uncertainty set is

$$\mathcal{U}_\sigma = \left\{ (L, H_1, \dots, H_{10}) : \begin{array}{l} L = \hat{L} + \Delta_L, \quad \|\Delta_L\|_2 \leq \varepsilon_L, \\ H_i = \hat{H}_i + \Delta H_i, \quad i = 1, \dots, 10, \\ \left(\sum_{i=1}^{10} \|\Delta H_i\|_2^2 \right)^{1/2} \leq \varepsilon_H \end{array} \right\}.$$

For each fitted model, we first test whether a common quadratic Lyapunov certificate holds for the linear part of every model in $\mathcal{U}_\sigma$. Using Petersen's lemma [18, 1], we solve

$$\begin{aligned} & \max_{P, \eta, \mu, t} \quad \eta \\ & \text{subject to} \quad P = P^\top, \quad P \succeq \delta I, \quad \text{tr}(P) = 10, \\ & \quad \quad \quad P \preceq tI, \quad \delta \leq t \leq 10, \\ & \quad \quad \quad \mu \geq 10^{-9}, \quad \eta \geq 0, \\ & \quad \quad \quad \begin{bmatrix} \hat{L}^\top P + P \hat{L} + \eta I + \mu \varepsilon_L^2 I & P \\ P & -\mu I \end{bmatrix} \preceq 0. \end{aligned} \tag{39}$$

with $\delta = 10^{-6}$. The quadratic uncertainty enters the local nonlinear certificate through $\gamma_{\text{rob}} = \hat{\gamma}_H + \varepsilon_H$, where $\hat{\gamma}_H = (\sum_{i=1}^{10} \|\hat{H}_i\|_2^2)^{1/2}$.

For a trial radius $r > 0$, the Petersen conditions are supplemented by

$$\eta \geq 2rt\gamma_{\text{rob}}.$$

The largest feasible radius is obtained by bisection with relative tolerance 10^{-4} . A certificate is accepted only if $\eta > 10^{-6}$, $\lambda_{\min}(P) \geq \delta - 10^{-8}$, the trace error is at most 10^{-5} , and the normalized LMI, upper-bound, and radius-constraint residuals are at most 10^{-5} .

Joint empirical coverage is the number of evaluation realizations for which both $e_L \leq \varepsilon_L$ and $e_H \leq \varepsilon_H$. It therefore evaluates whether the calibrated uncertainty bounds contain the known reference operators. This is distinct from robust certification, which asks whether the entire prescribed uncertainty set satisfies the sufficient Lyapunov condition. The evaluation results are summarized in Table 1.

Table 1. Coefficient-uncertainty and certification results for 100 independent evaluation realizations at each noise level.

Quantity	$\sigma = 0.005$	$\sigma = 0.020$
Empirical joint coverage	96/100	96/100
Hurwitz centers	100/100	100/100
Nominal Euclidean	0/100	0/100
Nominal P	100/100	100/100
Petersen P	100/100	0/100

The robust test gives different results at the two noise levels. At $\sigma = 0.005$, all 100 uncertainty sets are certified. At $\sigma = 0.020$, none is certified, although every fitted center remains Hurwitz and passes the nominal post-identification test. The difference, therefore, reflects the larger uncertainty set, not a change from stable to unstable fitted models.

5 Conclusion

This paper presented a post-identification certification workflow for sparse quadratic dynamical models. For each fitted model, identification is completed before certification, after which the coefficients and active terms are frozen. Predictive validation, Hurwitz screening, quadratic Lyapunov certification, local nonlinear certification, and robust certification then assess complementary aspects of model reliability. Together, these tests assess agreement with unseen data, local equilibrium stability, certified Lyapunov decay and invariant neighborhoods, and robustness to prescribed coefficient uncertainty.

The numerical results illustrate two mechanisms that can limit certification. For non-normal but Hurwitz dynamics, the fixed Euclidean metric can fail, while the post-identification SDP finds a quadratic Lyapunov metric for the same frozen dynamics. In the noisy ten-dimensional benchmark, all fitted model centers remained Hurwitz and nominally certifiable with the post-identification metric, whereas robust certification was lost for the larger calibrated uncertainty sets. Thus, nominal and robust certification provide distinct information about the identified model.

The reported certificates are sufficient and local. The derived radius is not an exact region of attraction, and failed certification does not prove instability. Although the numerical studies support the specific mechanisms examined here, they do not establish general applicability. Future work will examine larger power-system benchmarks and PMU measurements, and extend the framework to uncertainty in model structure.

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