

1. Introduction

Various types of systems (technical, industrial, economic, road transport, socio-political, etc.) can be viewed as conglomerates of complex cause-and-effect relations that link root causal events to final consequential events, often through intermediate events. Thus, a system's dynamics can be represented as a network of intertwined cause-to-effect sequences. If a system is analyzed in safety and reliability terms, observed events are inherently random, and the model of a system's behavior should be stochastic. An example of such a model is a Bayesian network (BN), widely used by safety and reliability analysts. It has the structure of a directed acyclic graph (DAG) with random variables assigned to its nodes, where links from parent to child variables signify direct cause-and-effect relationships. Each variable represents the set of events associated with a system component or an aspect of the system behavior. A variable's values correspond to individual events in the respective set. The variables are denoted as X_j , $j=1, \dots, n$, where n is the total number of network nodes. In addition to a random variable, each non-initial node is assigned a conditional probability table (CPT). It specifies how the variable in the given node depends on the variables in its parent nodes. The entries of a CPT for X_j are the conditional probabilities $\Pr[X_j=v_j \mid X_{\pi(j,k)}=v_{\pi(j,k)}, k=1, \dots, \lambda(j)]$, where $\pi(j,k)$ is the index of the k -th parent variable of X_j (according to the adopted ordering), $\lambda(j)$ is the number of X_j 's parents, and v_i and $v_{\pi(j,k)}$ are any values of X_i and $X_{\pi(j,k)}$. CPTs form the basis for all computations required in quantitative analysis of a BN.

There is abundant literature on the application of Bayesian networks to safety and reliability analysis. Introductory material on this topic can be found in Sucar 2021, Kjaerluff and Madsen 2013, and Koller and Friedman 2009. More advanced methods of their analysis and/or specialized software packages for that purpose are presented in Gansch et al. 2025, Mrowca et al. 2022, Scutari and Denis 2021, Yu et al. 2020, and Ding 2010. The issues of learning BN parameters and structure are considered in Fan et al. 2024, Kitson et al. 2023, Heckerman 2022, and Vázquez-Aguirre 2022. A recent comprehensive literature survey of literature on BNs in reliability assessment is contained in Wang et al. 2025.

This paper presents a new original method for computing the so-called forward and reverse conditional probabilities useful for detailed cause-effect analysis of BN-modeled systems. The first probability, in short FP, quantifies the chance of the collective occurrence of possible consequential events given that several causal events have taken place. It will be denoted as $\Pr[X_j=v_j, j \in E \mid X_j=v_j, j \in C]$, where C and E are the sets of causal and consequential events, and v_j is one of the values of X_j . Here, $\{X_j=v_j\}$ is the event that X_j assumes the value v_j . The second probability, in short RP, quantifies the chance of the collective occurrence of possible causal events given that several consequential events have taken place. It will be denoted as $\Pr[X_j=v_j, j \in C \mid X_j=v_j, j \in E]$, where the notation is explained above (note the swapped positions of C and E in FP and RP). As will be shown, FPs and RPs can be used to estimate the likelihoods of various cause-effect scenarios.

The rest of the paper is organized as follows. Section 2 contains notation and definitions with clarifying explanations. Section 3 presents a method to compute FPs and RPs, developed by the author of this paper. In Section 4, an illustrative example is given, which demonstrates the practical significance of these probabilities. Section 5 outlines the prospects for the further development of the presented method.

2. Basic definitions and notation

We consider a BN that models cause-and-effect relations in a certain system analyzed for safety and reliability. Variables with no parents are called initial. They represent root causes of all events that can occur in the considered system. We select several variables, not necessarily initial, that represent causes, and several non-initial ones that represent possible effects. We call the first causal and the second consequential. Throughout the paper, we will use the notation defined below.

$X < Y$: variable (node) X precedes variable Y in a BN, i.e. there exists a path from X to Y

U : set of selected causal variables; the variables in U can be both initial and non-initial

U' : subset of U composed of non-initial variables (U' can be empty, in which case U contains only initial variables)

$U^\wedge$: set of variables that precede those in U , but do not belong to U , i.e. $U^\wedge \cap U = \emptyset$

W : set of selected consequential variables; the variables in W are non-initial and preceded by those in U , i.e.

1) for each $X \in U$ there exists $Y \in W$ such that $X < Y$

2) for each $Y \in W$ there exists $X \in V$ such that $X < Y$

3) there do not exist $X \in U$ i $Y \in W$ such that $Y < X$

V : set of variables that precede those in W , but do not belong to U or W , i.e. $V \cap U = V \cap W = \emptyset$ (note that $U \cup V \subseteq W$)

$A, A', A^\wedge, B, C$: sets of indices of variables that respectively compose $U, U', U^\wedge, V, W$, i.e.

$U = \{X_i : i \in A\}, U' = \{X_i : i \in A'\}, U^\wedge = \{X_i : i \in A^\wedge\}, V = \{X_i : i \in B\}, W = \{X_i : i \in C\}$

$\lambda(j)$: number of parent variables of X_j

$\pi(j, k)$: index of the k -th parent variable of X_j (parents of X_j are ordered by increasing indices)

a_j : one of the values of X_j

$(a_j, j \in A)$: vector of arbitrary values of the variables $X_j, j \in A$; A can be replaced by $A', A^\wedge, B, C$

$Val(j)$: set of all values of X_j

$\times_{i \in A} Val(i)$: Cartesian product of $Val(i), i \in A$

$\Pr[X_j = v_j, j \in C \mid X_j = v_j, j \in A]$: forward probability

$\Pr[X_j = v_j, j \in A \mid X_j = v_j, j \in C]$: reverse probability

3. A method of computing FPs and RPs

Forward probabilities can be computed using the following

Lemma 1

It holds that

$$\Pr(X_i = a_i, i \in C \mid X_i = a_i, i \in A) = f/g, \quad (1)$$

where f and g are given by

$$f = \sum_{(a_i, i \in B) \in \times_{i \in B} Val(i)} \prod_{j \in C \cup B \cup A'} \Pr[X_j = a_j \mid X_{\pi(j, k)} = a_{\pi(j, k)}, 1 \leq k \leq \lambda(j)] \quad (2)$$

$$g = \sum_{(a_i, i \in A^\wedge) \in \times_{i \in A^\wedge} Val(i)} \prod_{j \in A' \cup A^\wedge} \Pr[X_j = a_j \mid X_{\pi(j, k)} = a_{\pi(j, k)}, 1 \leq k \leq \lambda(j)] \quad (3)$$

In (2), the summation runs over all vectors $(a_i, i \in B)$ belonging to the Cartesian product of the sets $Val(i), i \in B$. If $B = \emptyset$, then the sum in Eq. (2) is unspecified and

$$f = \prod_{j \in C \cup A'} \Pr(X_j = a_j | X_{\pi(j,k)} = a_{\pi(j,k)}, 1 \leq k \leq \lambda(j)) \quad (4)$$

In (3) we sum over all vectors $(a_i, i \in B)$ that belong to the Cartesian product of the sets $Val(i), i \in A^\wedge$. If $A' = \emptyset$ (U contains only initial variables), then $A^\wedge = \emptyset$ and the sum in Eq. (3) is unspecified. In this case, the denominator in Eq. (1) equals one, i.e. $g=1$. Further, if $A' \neq \emptyset$ and $A^\wedge = \emptyset$, then this denominator is given by

$$g = \prod_{j \in A'} \Pr(X_j = a_j | X_{\pi(j,k)} = a_{\pi(j,k)}, 1 \leq k \leq \lambda(j)) \quad (5)$$

Remark 1: probabilities on the right-hand side of Eqs. (2)-(4) are stored in the CPTs of the variables X_j , with j in the respective set of indices, hence applying Eqs. (2)-(4) requires that these CPTs be given.

Remark 2: If any of the variables $X_j, j \in B$, in Eq. (2), or any of the variables $X_j, j \in A^\wedge$, in Eq. (3) is initial, the respective probability is non-conditional, i.e.

$$\Pr(X_j = a_j | X_{\pi(j,k)} = a_{\pi(j,k)}, 1 \leq k \leq \lambda(j)) = \Pr(X_j = a_j) \quad (6)$$

Eq. (6) follows from the assumed mutual independence of the initial variables. This assumption is justified as follows: if a variable X depends on some initial variables, then X has at least one parent, so X is not initial. Here, we postulate that a dependence relation involves a link or path in a BN between dependent variables. Additionally, $\lambda(j)=0$ if X_j is initial, so the conditional probability on the left-hand side of Eq. (6) is then unspecified.

Since Eqs. (2)-(6) are somewhat complicated, Lemma 1 will be illustrated using the network in Fig. 1. For this purpose, let us put $U=\{X_1, X_2\}$, $W=\{X_5, X_9\}$, and compute the forward probability $\Pr(X_5=a_5, X_9=a_9 | X_1=a_1, X_2=a_2)$. We have: $A=\{1,2\}$, $U'=U^\wedge=\emptyset$, $A'=A^\wedge=\emptyset$, $V=\{X_3, X_4, X_7, X_8\}$, $B=\{3,4,7,8\}$, $C=\{5,9\}$. Since A' is empty, it holds that $g=1$, and FP is computed from Eq. (2), which yields:

$$\begin{aligned}
& \Pr(X_5 = a_5, X_9 = a_9 \mid X_1 = a_1, X_2 = a_2) = \\
& \quad \Pr(X_5 = a_5 \mid X_1 = a_1, X_2 = a_2, X_3 = a_3) \\
& \quad \times \Pr(X_9 = a_9 \mid X_7 = a_7, X_8 = a_8) \\
& \quad \times \Pr(X_3 = a_3) \\
& = \sum_{\substack{(a_3, a_4, a_7, a_8) \\ \in \text{Val}(3) \times \text{Val}(4) \\ \times \text{Val}(7) \times \text{Val}(8)}}} \Pr(X_4 = a_4 \mid X_1 = a_1, X_2 = a_2) \\
& \quad \times \Pr(X_7 = a_7 \mid X_4 = a_4) \\
& \quad \times \Pr(X_8 = a_8 \mid X_5 = a_5, X_7 = a_7)
\end{aligned} \tag{7}$$

Note that X_3 , which belongs to V , is an initial variable, so the third factor in the product in Eq. (7) is a non-conditional probability. Also note that X_1, X_2, X_5, X_9 have fixed values equal to a_1, a_2, a_5, a_9 respectively.

As follows from Eq. (7), to compute $\Pr(X_5=a_5, X_9=a_9 \mid X_1=a_1, X_2=a_2)$ we need the CPTs of X_5, X_9, X_4, X_7, X_8 , and the probabilities $\Pr(X_3=a), a \in \text{Val}(3)$, i.e. the distribution of the initial variable X_3 . They can be obtained from statistical data (if available) or from other sources, e.g. experts' elicitations.

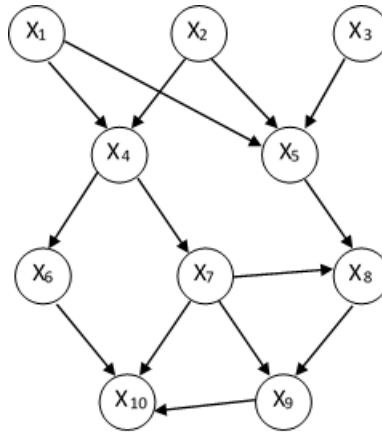


Fig. 1. An example of a Bayesian network

Now we turn to reverse probabilities, which can be found from the equations given by the second lemma.

Lemma 2

It holds that

$$\Pr(X_i = a_i, i \in A | X_i = a_i, i \in C) = f/g, \quad (8)$$

where f and g are given by

$$f = \sum_{(a_i, i \in B) \in \times_{i \in B} \text{Val}(i)} \prod_{j \in C \cup B \cup A} \Pr(X_j = a_j | X_{\pi(j,k)} = a_{\pi(j,k)}, 1 \leq k \leq \lambda(j)) \quad (9)$$

$$g = \sum_{(a_i, i \in B \cup A) \in \times_{i \in B \cup A} \text{Val}(i)} \prod_{j \in C \cup B \cup A} \Pr(X_j = a_j | X_{\pi(j,k)} = a_{\pi(j,k)}, 1 \leq k \leq \lambda(j)) \quad (10)$$

If $B = \emptyset$, then

$$f = \prod_{j \in C \cup A} \Pr(X_j = a_j | X_{\pi(j,k)} = a_{\pi(j,k)}, 1 \leq k \leq \lambda(j)) \quad (11)$$

Remark 1: Only the sets A, B, C (corresponding to U, V, W) appear in Eqs. (9)-(11). The sets A' and $A^\wedge$ (corresponding to U' and $U^\wedge$) are not present.

Remark 2: The components of sums in Eqs. (9) and (10) are identical, but the ranges of summation differ.

Remark 3: If any of the variables $X_j, j \in B \cup A$, in Eqs. (9)-(11) is initial, then the second remark to Lemma 1 applies, i.e. Eq. (6) holds.

To illustrate Eqs. (9)-(11), let us again refer to Fig. 1 and compute the reverse probability $\Pr(X_1=a_1, X_2=a_2 | X_5=a_5, X_9=a_9)$. As previously, $A=\{1,2\}$, $B=\{3,4,7,8\}$, $C=\{5,9\}$. From Eqs. (8)-(11), we obtain:

$$\begin{aligned} & \Pr(X_5 = a_5 | X_1 = a_1, X_2 = a_2, X_3 = a_3) \\ & \quad \times \Pr(X_9 = a_9 | X_7 = a_7, X_8 = a_8) \\ & \quad \times \Pr(X_3 = a_3) \\ f = \sum_{\substack{(a_3, a_4, a_7, a_8) \\ \in \text{Val}(3) \times \text{Val}(4) \\ \times \text{Val}(7) \times \text{Val}(8)}}} & \quad \times \Pr(X_4 = a_4 | X_1 = a_1, X_2 = a_2) \\ & \quad \times \Pr(X_7 = a_7 | X_4 = a_4) \\ & \quad \times \Pr(X_8 = a_8 | X_5 = a_5, X_7 = a_7) \\ & \quad \times \Pr(X_1 = a_1) \times \Pr(X_2 = a_2) \end{aligned} \quad (12)$$

$$\begin{aligned}
& \Pr(X_5 = a_5 \mid X_1 = a_1, X_2 = a_2, X_3 = a_3) \\
& \quad \times \Pr(X_9 = a_9 \mid X_7 = a_7, X_8 = a_8) \\
& \quad \quad \times \Pr(X_3 = a_3) \\
g = \sum_{\substack{(a_3, a_4, a_7, a_8, a_1, a_2) \\ \in Val(3) \times Val(4) \\ \times Val(7) \times Val(8) \\ \times Val(1) \times Val(2)}}} & \quad \times \Pr(X_4 = a_4 \mid X_1 = a_1, X_2 = a_2) \\
& \quad \quad \times \Pr(X_7 = a_7 \mid X_4 = a_4) \\
& \quad \quad \times \Pr(X_8 = a_8 \mid X_5 = a_5, X_7 = a_7) \\
& \quad \quad \times \Pr(X_1 = a_1) \times \Pr(X_2 = a_2)
\end{aligned} \tag{13}$$

$$\Pr(X_1 = a_1, X_2 = a_2 \mid X_5 = a_5, X_9 = a_9) = f/g \tag{14}$$

Note the different ranges of summation in Eqs. (12) and (13). Also note that X_3 from the set V , and X_1, X_2 from the set U are initial variables, so the respective factors of the products in Eqs. (12) and (13) are non-conditional probabilities. Last but not least, mind that X_1, X_2, X_5, X_9 have fixed values equal to a_1, a_2, a_5, a_9 respectively.

[Proofs of Lemmas 1 and 2 will appear in the extended version of this paper, which is currently prepared for publication.](#) Let us recall that the formulas from Lemmas 1 and 2 allow us to compute the probability of the likely occurrence of selected consequential (causal) events, given that particular causal (consequential) events have taken place. In the first case, the consequential events are supposed effects of the given causal events, while in the second case, the causal events are the supposed causes of the given consequential events.

Based on the obtained results, the following two algorithms for computing FPs and RPs have been constructed.

Algorithm 1

Select the sets of indices C and A from the set $\{1, \dots, n\}$;

Find the sets A' , $A^\wedge$, B , based on C and A ;

$f \leftarrow 0$; $g \leftarrow 0$;

If $B \neq \emptyset$, then for each vector $(a_i, i \in B)$ in $\times_{i \in B} Val(i)$ do:

$$f \leftarrow f + \prod_{j \in C \cup B \cup A'} \Pr(X_j = a_j | X_{\pi(j,k)} = a_{\pi(j,k)}, 1 \leq k \leq \lambda(j));$$

If $B = \emptyset$, then

$$f \leftarrow \prod_{j \in C \cup A'} \Pr(X_j = a_j | X_{\pi(j,k)} = a_{\pi(j,k)}, 1 \leq k \leq \lambda(j));$$

If $A' \neq \emptyset$ and $A^\wedge \neq \emptyset$, then for each vector $(a_i, i \in A^\wedge)$ in $\times_{i \in A^\wedge} Val(i)$ do:

$$g \leftarrow g + \prod_{j \in A' \cup A^\wedge} \Pr(X_j = a_j | X_{\pi(j,k)} = a_{\pi(j,k)}, 1 \leq k \leq \lambda(j));$$

If $A' \neq \emptyset$ and $A^\wedge = \emptyset$, then

$$g \leftarrow \prod_{j \in A'} \Pr(X_j = a_j | X_{\pi(j,k)} = a_{\pi(j,k)}, 1 \leq k \leq \lambda(j));$$

If $A' = \emptyset$, then $g \leftarrow 1$;

$FP \leftarrow f / g$.

Algorithm 2

Select the sets of indices C and A from the set $\{1, \dots, n\}$;

Find the set B , based on C and A ;

$f \leftarrow 0$; $g \leftarrow 0$;

If $B \neq \emptyset$, then for each vector $(a_i, i \in B)$ in $\times_{i \in B} Val(i)$ do:

$$f \leftarrow f + \prod_{j \in C \cup B \cup A} \Pr(X_j = a_j | X_{\pi(j,k)} = a_{\pi(j,k)}, 1 \leq k \leq \lambda(j));$$

If $B = \emptyset$, then

$$f \leftarrow \prod_{j \in C \cup A} \Pr(X_j = a_j | X_{\pi(j,k)} = a_{\pi(j,k)}, 1 \leq k \leq \lambda(j));$$

For each vector $(a_i, i \in B \cup A)$ in $\times_{i \in B \cup A} Val(i)$ do:

$$g \leftarrow g + \prod_{j \in C \cup B \cup A} \Pr(X_j = a_j | X_{\pi(j,k)} = a_{\pi(j,k)}, 1 \leq k \leq \lambda(j));$$

$RP \leftarrow f / g$.

Remark: Left arrow denotes the assignment operation.

4. A practical example of using FPs and RPs

The use of FPs and RPs to express various cause-and-effect relations will now be illustrated with an example adapted from Malinowski 2025. The figure below is a fragment of the BN that models certain safety aspects of a biogas digester.

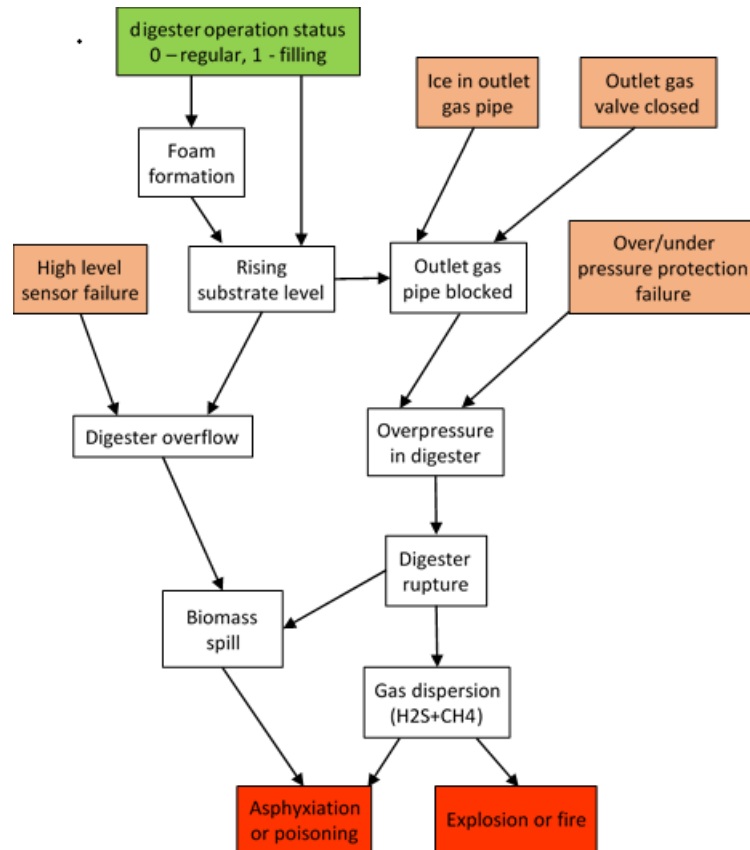


Fig. 2. BN for safety analysis of a biogas digester. Adapted from Malinowski, 2025.

Random variables are assigned to the individual nodes as follows:

X_1 – digester operation status

X_2 – high level sensor failure

X_3 – ice in outlet gas pipe

X_4 – outlet gas valve closed

X_5 – over/under pressure protection failure

X_6 – foam formation

X_7 – rising substrate level

X_8 – outlet gas pipe blocked

X_9 – digester overflow

X_{10} – overpressure in digester

X_{11} – biomass spill

X_{12} – digester rupture

X_{13} – gas dispersion

X_{14} – poisoning or asphyxiation

X_{15} – explosion or fire

All these RVs are binary, where 1 denotes an occurrence of the respective event and 0 its non-occurrence. If, for instance, we want to compute the probability of a biomass spill without explosion or fire outbreak, provided that the high level sensor is functioning, and the outlet gas pipe is ice-clogged or the outlet gas valve is closed, this probability is given by

$$\sum_{i=1}^3 \Pr(X_{11} = 1, X_{15} = 0 \mid E_i) \Pr(E_i) / \sum_{i=1}^3 \Pr(E_i) \quad (15)$$

where

$$E_1 = \{X_2 = 0, X_3 = 0, X_4 = 1\},$$

$$E_2 = \{X_2 = 0, X_3 = 1, X_4 = 0\},$$

$$E_3 = \{X_2 = 0, X_3 = 1, X_4 = 1\} \quad (16)$$

The conditional probabilities in the numerator in Eq. (15) are FPs that can be calculated using Lemma 1. The probabilities $\Pr(E_i)$ are easy to compute, because E_1, E_2, E_3 are conjunctions of independent initial events, e.g.

$$\Pr(E_1) = \Pr(X_2=0)\Pr(X_3=0)\Pr(X_4=1) \quad (17)$$

Now assume that we want to compute the probability that the ice-clogged outlet gas pipe is a likely cause of the occurred substrate spill. This is the reverse probability $\Pr(X_3=1 \mid X_{11}=1)$ that can be calculated using Lemma 2. As shown, appropriately defined FPs and RPs are essential for a detailed probabilistic analysis of various cause-and-effect scenarios in BN-modeled systems.

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