BIPOLAR SORTING AND RANKING OF MULTISTAGE ALTERNATIVES

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1. Introduction

1.1. Classical Bipolar approach

The classical Bipolar method, proposed by E. Konarzewska-Gubała is an MCDA method. The individual phases of this method use elements of the Electre methodology. as well as algorithms of confrontation

A fundamental feature of the classic Bipolar method is that the decision alternatives are not compared directly with each other, but by means of two **sets of reference points**: objects with desired characteristics, called "**good**" objects, and objects with undesired characteristics, called "**bad**" objects.

Phases of the classical Bipolar Method

Phase I. . Decision alternatives ***are compared*** with good and bad objects. I

Phase *II*. *The position* of each alternative ***with respect to the bipolar reference system*** is established.

Phase III. The alternatives are ***classified: first separately***, with respect to the good and bad reference sets, ***then jointly***. The alternatives are divided into indexed classes so that each alternative from a lower-indexed class is preferred over any alternative from a higher-indexed class. Within each class, a linear ordering is defined.

1.2. Extension to multistage decision processes

Almost simultaneously with the introduction of the classic Bipolar method the issue of a possible extension of this approach to the analysis of multistage, multicriteria decision processes arose. That attempt, however, had not been entirely successful and research in this direction was discontinued.

Now I am trying to tackle this problem again and present possible applications of an essential fragment of the classic Bipolar approach – which is a single-stage procedure – to control multistage discrete decision processes.

This requires that new notions be defined, directly related to the extension being constructed, such as stage alternative, multistage alternative, stage reference sets, or the importance of the criteria in the consecutive process stages.

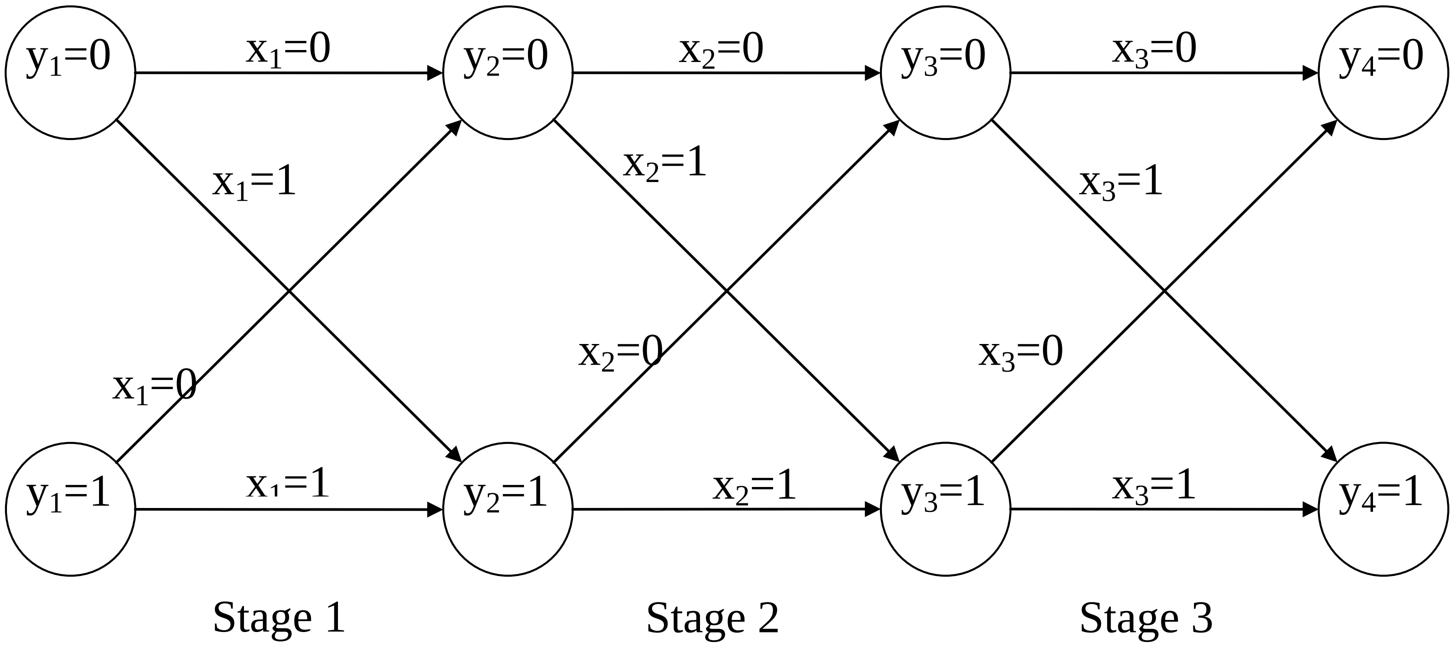
2. Assumptions and notation

Let us start from the following example.

We consider a three-stage decision process. The sets of feasible states and decisions are as follows:

$$\mathbf{Y}_t = \{0,1\} \qquad \text{for } t = 1,\ldots,4,$$

$$\mathbf{X}_t(0) = \{0, 1\},\ \mathbf{X}_t(1) = \{0, 1\} \qquad \text{for } t = 1, 2, 3.$$



Notation

T – the number of process stages ($t = 1,\ldots,T$),

$\mathbf{Y}_t$ – the set of feasible states at the beginning of stage t ($y_t \in \mathbf{Y}_t$)

$\mathbf{X}_t(y_t)$ – the set of feasible decisions at the beginning of stage t for state y_t ($x_t \in \mathbf{X}_t(y_t)$)

Ω_t – the transition function for stage t. We have: $y_{t+1} = x_t$,

$\mathbf{A}_t$ – the set of ***stage alternatives*** for stage t ($\mathbf{a}_t \in \mathbf{A}_t$, $\mathbf{a}_t = (y_t, x_t) = (y_t, y_{t+1})$)

$\mathbf{A}$ – the set of all ***multistage alternatives*** $\mathbf{a} \in \mathbf{A}$, $\mathbf{a} = (\mathbf{a}_1,\ldots,\mathbf{a}_T) = ((y_1, x_1),\ldots,y_T, x_T)) = (y_1,\ldots,y_{T+1})$

$\mathbf{G}_t$ – the stage set of good objects,

$\mathbf{B}_t$ – the stage set of bad objects,

$\mathbf{R}_t = (\mathbf{G}_t, \mathbf{B}_t)$ – the stage reference system ($\mathbf{G}_t \cap \mathbf{B}_t = \varnothing$)

$\mathbf{R} = (\mathbf{R}_1, \ldots \mathbf{R}_T)$ – the multistage reference system

K – the number of the all the criteria considered ($k = 1,\ldots,K$)

$\mathbf{C}_t$ – the set of stage criteria (c_t^k – k-th criterion at stage t)

f_t^k – the stage criterion function for stage t (f_t^k: $\mathbf{A}_t \cup \mathbf{R}_t \rightarrow \boldsymbol{K}^k$ for $k = 1, \ldots, K$,

and $\boldsymbol{K}_k$ is a cardinal, ordinal or binary scale). We assume, that

$$f_t^k(\mathbf{b}_t) < f_t^k(\mathbf{g}_t)$$

w_t^k – the weight of the relative importance of criterion k in stage t ($\sum_{k=1}^K w_t^k = 1$, $\forall_{k=1,\ldots,K}\ \ w_t^k \geq 0$)

3. Stage alternatives

3.1. Comparison of stage alternatives with stage reference objects

The comparison of the values $f_t^k(\mathbf{a}_t)$ and $f_t^k(\mathbf{r}_t)$ can result in one of the following situations:

$$f_t^k(\mathbf{a}_t) > f_t^k(\mathbf{r}_t) \tag{1}$$

$$f_t^k(\mathbf{a}_t) = f_t^k(\mathbf{r}_t) \tag{2}$$

$$f_t^k(\mathbf{a}_t) < f_t^k(\mathbf{r}_t \tag{3}$$

<u>Step 1:</u> 0-1 indicators

$$\varphi_t^{k+}(\mathbf{a}_t, \mathbf{r}_t) = \begin{cases} 1, & \text{if } f_t^k(\mathbf{a}_t) - f_t^k(\mathbf{r}_t) > 0 \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

$$\varphi_t^{k=}(\mathbf{a}_t, \mathbf{r}_t) = \begin{cases} 1, & \text{if } f_t^k(\mathbf{a}_t) = f_t^k(\mathbf{r}_t) \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

$$\varphi_t^{k-}(\mathbf{a}_t, \mathbf{r}_t) = \begin{cases} 1, & \text{if } f_t^k(\mathbf{a}_t) - f_t^k(\mathbf{r}_t) < 0 \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

Step 2: Stage indicators

$$c_t^{+}\left(\mathbf{a}_t,\mathbf{r}_t\right)=\sum_{k=1}^{n} w_t^k \varphi_t^{k+}\left(\mathbf{a}_t,\mathbf{r}_t\right), \quad (7)$$

$$c_t^{=}\left(\mathbf{a}_t,\mathbf{r}_t\right)=\sum_{k=1}^{n} w_t^k \varphi_t^{k=}\left(\mathbf{a}_t,\mathbf{r}_t\right), \quad (8)$$

$$c_t^{-}\left(\mathbf{a}_t,\mathbf{r}_t\right)=\sum_{k=1}^{n} w_t^k \varphi_t^{k-}\left(\mathbf{a}_t,\mathbf{r}_t\right). \quad (9)$$

Step 3: Outranking stage indicators

Step 3a.

If

$$c_t^{+}(\mathbf{a}_t, \mathbf{r}_t) > c_t^{-}(\mathbf{a}_t, \mathbf{r}_t), \quad (10)$$

then the stage alternative $\mathbf{a}_t \in \mathbf{A}_t$ outranks the stage reference object $\mathbf{r}_t \in \mathbf{R}_t$ and

$$d_t^{+}(\mathbf{a}_t, \mathbf{r}_t) = c_t^{+}(\mathbf{a}_t, \mathbf{r}_t) + c_t^{=}(\mathbf{a}_t, \mathbf{r}_t), \quad d_t^{-}(\mathbf{a}_t, \mathbf{r}_t) = 0. \ (14) \quad (11)$$

If

$$c_t^{+}(\mathbf{a}_t, \mathbf{r}_t) < c_t^{-}(\mathbf{a}_t, \mathbf{r}_t), \quad (12)$$

the reference object $\mathbf{r}_t \in \mathbf{R}_t$ outranks the stage alternative $\mathbf{a}_t \in \mathbf{A}_t$ and

$$d_t^{+}(\mathbf{a}_t, \mathbf{r}_t) = 0, \qquad d_t^{-}(\mathbf{a}_t, \mathbf{r}_t) = c_t^{-}(\mathbf{a}_t, \mathbf{r}_t) + c_t^{=}(\mathbf{a}, \mathbf{r}_t). \quad (13)$$

Step 3b

If

$$c_t^{+}(\mathbf{a}_t, \mathbf{r}_t) = c_t^{-}(\mathbf{a}_t, \mathbf{r}_t), \tag{14}$$

the stage alternative $\mathbf{a}_t \in \mathbf{A}_t$ is evaluated as equally good as the reference object $\mathbf{r}_t \in \mathbf{R}_t$ and

- if $\mathbf{r}_t$ is a good object, then

$$d_t^{+}(\mathbf{a}_t, \mathbf{r}_t) = c_t^{+}(\mathbf{a}_t, \mathbf{r}_t) + c_t^{=}(\mathbf{a}_t, \mathbf{r}_t), \qquad d_t^{-}(\mathbf{a}_t, \mathbf{r}_t) = 0, \tag{15}$$

- if $\mathbf{r}_t$ is a bad object, then

$$d_t^{+}(\mathbf{a}_t, \mathbf{r}_t) = 0, \qquad d_t^{-}(\mathbf{a}_t, \mathbf{r}_t) = c_t^{+}(\mathbf{a}_t, \mathbf{r}_t) + c_t^{=}(\mathbf{a}_t, \mathbf{r}_t). \tag{16}$$

Step 4. Stage relationships:

Relation of stage preference

$$\mathbf{a}_t \ \mathrm{L_t} \ \mathbf{r}_t \ \text{ iff } \ c_t^{+}(\mathbf{a}_t, \mathbf{r}_t) \geq c_t^{-}(\mathbf{a}_t, \mathbf{r}_t), \tag{17}$$

$$\mathbf{r}_t \ \mathrm{L_t} \ \mathbf{a}_t \ \text{ iff } \ c_t^{+}(\mathbf{a}_t, \mathbf{r}_t) \leq d_t^{-}(\mathbf{a}_t, \mathbf{r}_t), \tag{18}$$

Relation of stage indifference

$$\mathbf{a}_t \ \mathrm{I_t} \ \mathbf{r}_t \ \text{ iff } \ c_t^{+}(\mathbf{a}_t, \mathbf{r}_t) = c_t^{-}(\mathbf{a}_t, \mathbf{r}_t). \tag{19}$$

3.2. Position of a stage alternative with respect to the bipolar stage reference system

The position of the stage alternative $\mathbf{a}_t$ with respect to the stage reference set of bad objects

$$\mathcal{L}_t(\mathbf{a}_t, \mathbf{G}_t) = \{h: \mathbf{a}_t \, \mathrm{L}_t \, \mathbf{g}_t^{(h)}, \mathbf{g}_t^{(h)} \in \mathbf{G}_t\}, \tag{20}$$

$$\mathcal{L}_t(\mathbf{G}_t, \mathbf{a}_t) = \{h: \mathbf{g}_t^{(h)} \, \mathrm{L}_t \, \mathbf{a}_t, \mathbf{g}_t^{(h)} \in \mathbf{G}_t\}, \tag{21}$$

$$\mathcal{I}_t(\mathbf{a}_t, \mathbf{G}_t) = \{h: \mathbf{a}_t \, \mathrm{I}_t \, \mathbf{g}_t^{(h)}, \mathbf{g}_t^{(h)} \in \mathbf{G}_t\}. \tag{22}$$

Case S1

If

$$\mathcal{L}_t(\mathbf{a}_t, \mathbf{G}_t) \cup \mathcal{I}_t(\mathbf{a}_t, \mathbf{G}_t) \neq \varnothing. \tag{23}$$

then

$$d_{\mathbf{G}}^{+}(\mathbf{a}_t) = \max\{d_t^{+}(\mathbf{a}_t, \mathbf{g}_t^{(h)}): h \in \mathcal{L}_t(\mathbf{a}_t, \mathbf{G}_t) \cup \mathcal{I}_t(\mathbf{a}_t, \mathbf{G}_t)\}, \qquad d_{\mathbf{G}}^{-}(\mathbf{a}_t) = 0. \tag{24}$$

Case S2

If

$$\mathcal{L}_s(\mathbf{a}_t, \mathbf{G}_t) \cup \mathcal{I}_t(\mathbf{a}_t, \mathbf{G}_t) = \varnothing \wedge \mathcal{L}_s(\mathbf{G}_t, \mathbf{a}_t) \neq \varnothing. \tag{25}$$

then

$$d_{\mathbf{G}}^{+}(\mathbf{a}_t) = 0, \qquad d_{\mathbf{G}}^{-}(\mathbf{a}_t) = \min\{d_t^{+}(\mathbf{a}_t, \mathbf{g}_t^{(h)}): h \in \mathcal{L}_t(\mathbf{a}_t, \mathbf{G}_t) \cup \mathcal{I}_t(\mathbf{a}_t, \mathbf{G}_t)\}. \tag{26}$$

The position of the stage alternative $\mathbf{a}_t$ with respect to the stage reference set of bad objects

$$\mathcal{L}_t(\mathbf{a}_t , \mathbf{B}_t) = \{h: \mathbf{a}_t \, \mathrm{L}_t \, \mathbf{b}^{(h)}, \mathbf{b}_t^{(h)} \in \mathbf{B}_t\}, \tag{27}$$

$$\mathcal{L}_t(\mathbf{B}_t , \mathbf{a}_t) = \{h: \mathbf{b}_\mathbf{t}^{(h)} \, \mathrm{L}_t \, \mathbf{a}_t, \mathbf{b}_t^{(h)} \in \mathbf{B}_t\}, \tag{28}$$

$$\mathcal{I}_t (\mathbf{B}_t , \mathbf{a}_t) = \{h: \mathbf{b}_\mathbf{t}^{(h)} \, \mathrm{I}_t \, \mathbf{a}_t , \mathbf{b}_t^{(h)} \in \mathbf{B}_t\}. \tag{29}$$

Case F1.

If

$$\mathcal{L}_t (\mathbf{B}_t , \mathbf{a}_\mathbf{t}) \cup \mathcal{I}_t (\mathbf{B}_t , \mathbf{a}_\mathbf{t}) = \varnothing \wedge \mathcal{L}_t(\mathbf{a}_\mathbf{t} , \mathbf{B}_t) \neq \varnothing. \tag{30}$$

then

$$d_\mathbf{B}^+(\mathbf{a}_t) = \min \{d_t^+(\mathbf{a}_t, \mathbf{b}_t^{(h)}): h \in \mathcal{L}_s(\mathbf{a}_t , \mathbf{B}_t)\}, \qquad d_\mathbf{B}^-(\mathbf{a}_t) = 0. \tag{31}$$

Case F2.

If

$$\mathcal{L}_t(\mathbf{B}_t, \mathbf{a}_t) \cup \mathcal{I}_t (\mathbf{B}_t, \mathbf{a}_t) \neq \varnothing. \tag{32}$$

then

$$d_\mathbf{B}^+(\mathbf{a}_t) = 0, \qquad d_\mathbf{B}^-(\mathbf{a}_t) = \max \{d_t^-(\mathbf{a}_t, \mathbf{b}_t^{(h)}): h \in \mathcal{L}_t(\mathbf{B}_t, \mathbf{a}_t) \cup \mathcal{I}_t(\mathbf{B}_t , \mathbf{a}_t)\}. \tag{33}$$

4. Relationships in the set of multistage alternatives

Multistage success achievement degrees:

$$\mathrm{d}_{\mathbf{G}}^{+}\left(\mathbf{a}\right)=\frac{1}{T}\sum_{t=1}^{T}\mathrm{d}_{\mathbf{G}}^{+}\left(\mathbf{a}_{t}\right) \quad (34)$$

$$\mathrm{d}_{\mathbf{G}}^{-}\left(\mathbf{a}\right)=\frac{1}{T}\sum_{t=1}^{T}\mathrm{d}_{\mathbf{G}}^{-}\left(\mathbf{a}_{t}\right). \quad (35)$$

Multistage failure avoidance degrees:

$$\mathrm{d}_{\mathbf{B}}^{+}\left(\mathbf{a}\right)=\frac{1}{T}\sum_{t=1}^{T}\mathrm{d}_{\mathbf{B}}^{+}\left(\mathbf{a}_{t}\right) \quad (36)$$

$$\mathrm{d}_{\mathbf{B}}^{-}\left(\mathbf{a}\right)=\frac{1}{T}\sum_{t=1}^{T}\mathrm{d}_{\mathbf{B}}^{-}\left(\mathbf{a}_{t}\right) \quad (37)$$

$$\mathrm{d}(\mathbf{a}) = [\mathrm{d}_{\mathbf{G}}^{+}(\mathbf{a}), \mathrm{d}_{\mathbf{G}}^{-}(\mathbf{a}), \mathrm{d}_{\mathbf{B}}^{+}(\mathbf{a}), \mathrm{d}_{\mathbf{B}}^{-}(\mathbf{a})]. \quad (38)$$

4.1. Sorting multistage alternatives

$$\mathbf{A}^1 = \{\mathbf{a} \in \mathbf{A}: \quad d_{\mathbf{G}}^{+}(\mathbf{a}) > 0, \; d_{\mathbf{G}}^{-}(\mathbf{a}) = 0, \; d_{\mathbf{B}}^{+}(\mathbf{a}) > 0, \; d_{\mathbf{B}}^{-}(\mathbf{a}) = 0\} \tag{39}$$

$\mathbf{A}^1$ constitutes the class of the best alternatives.

$$\mathbf{A}^2 = \{a \in \mathbf{A}: \quad d_{\mathbf{G}}^{+}(\mathbf{a}) > 0, \; d_{\mathbf{G}}^{-}(\mathbf{a}) > 0, \; d_{\mathbf{B}}^{+}(\mathbf{a}) > 0 \; d_{\mathbf{B}}^{-}(\mathbf{a}) = 0\} \tag{40}$$

Multistage alternatives from $\mathbf{A}^2$ are evaluated lower than those from $\mathbf{A}^1$.

$$\mathbf{A}^3 = \{a \in \mathbf{A}: \quad d_{\mathbf{G}}^{+}(\mathbf{a}) > 0, \; d_{\mathbf{G}}^{-}(\mathbf{a}) > 0, \; d_{\mathbf{B}}^{+}(\mathbf{a}) > 0, \; d_{\mathbf{B}}^{-}(\mathbf{a}) > 0\}. \tag{41}$$

Multistage alternatives from $\mathbf{A}^3$ are evaluated lower than those from $\mathbf{A}^2$.

$$\mathbf{A}^4 = \{a \in \mathbf{A}: \quad d_{\mathbf{G}}^{+}(\mathbf{a}) = 0, \; d_{\mathbf{G}}^{-}(\mathbf{a}) > 0, \; d_{\mathbf{B}}^{+}(\mathbf{a}) > 0, \; d_{\mathbf{B}}^{-}(\mathbf{a}) = 0\} \tag{42}$$

Multistage alternatives from $\mathbf{A}^4$ are evaluated lower than those from $\mathbf{A}^3$.

$$\mathbf{A}^5 = \{a \in \mathbf{A}: \quad d_{\mathbf{G}}^{+}(\mathbf{a}) = 0, \; d_{\mathbf{G}}^{-}(\mathbf{a}) > 0, \; d_{\mathbf{B}}^{+}(\mathbf{a}) > 0 \; d_{\mathbf{B}}^{-}(\mathbf{a}) > 0\} \tag{43}$$

Multistage alternatives from $\mathbf{A}^5$ are evaluated lower than those from $\mathbf{A}^4$.

$$\mathbf{A}^6 = \{a \in \mathbf{A}: \quad d_{\mathbf{G}}^{+}(\mathbf{a}) = 0, \quad d_{\mathbf{G}}^{-}(\mathbf{a}) > 0, \quad d_{\mathbf{B}}^{+}(\mathbf{a}) = 0, \quad d_{\mathbf{B}}^{-}(\mathbf{a}) > 0\} \tag{44}$$

Multistage alternatives from $\mathbf{A}^6$ are evaluated lower than those from $\mathbf{A}^5$.

$\mathbf{A}^7, \mathbf{A}^8, \mathbf{A}^9, \mathbf{A}^{10}, \mathbf{A}^{11}, \mathbf{A}^{12}, \mathbf{A}^{13}, \mathbf{A}^{14}, \mathbf{A}^{15}, \mathbf{A}^{16}$

$$\mathbf{A}^1 \cup \mathbf{A}^2 \cup \ldots \cup \mathbf{A}^{16} = \mathbf{A} \tag{45}$$

Because of the construction of these classes, we have:

$$\mathbf{A}^1 \cap \mathbf{A}^2 \cap \ldots \cap \mathbf{A}^{16} = \emptyset \tag{46}$$

Our assumptions easily lead to the conclusion that:

$$\mathbf{A}^i = \emptyset \tag{47}$$

for $i = 7,\ldots,16$.

Therefore, the multistage alternatives can be sorted into the six classes $\mathbf{A}^1,\ldots,\mathbf{A}^6$ and

$$\mathbf{A}^1 \cup \mathbf{A}^2 \cup \mathbf{A}^3 \cup \mathbf{A}^4 \cup \mathbf{A}^5 \cup \mathbf{A}^6 = \mathbf{A} \tag{48}$$

If $k < l$, then each multistage alternative from class $\mathbf{A}^k$ ***is preferred*** over any multistage alternative from class $\mathbf{A}^l$.

4.2. Ranking the multistage alternatives

Let

$$\delta(\mathbf{a}^{(i)}) = d_G^{+}(\mathbf{a}^{(i)}) - d_G^{-}(\mathbf{a}^{(i)}) + d_B^{+}(\mathbf{a}^{(i)}) - d_B^{-}(\mathbf{a}^{(i)}) \qquad .(49)$$

We order the alternatives within the classes:

$$\mathbf{a}^{(i)} \text{ is preferred to } \mathbf{a}^{(j)} \text{, iff} \quad \delta(\mathbf{a}^{(i)}) > \delta(\mathbf{a}^{(j)}) \qquad (50)$$

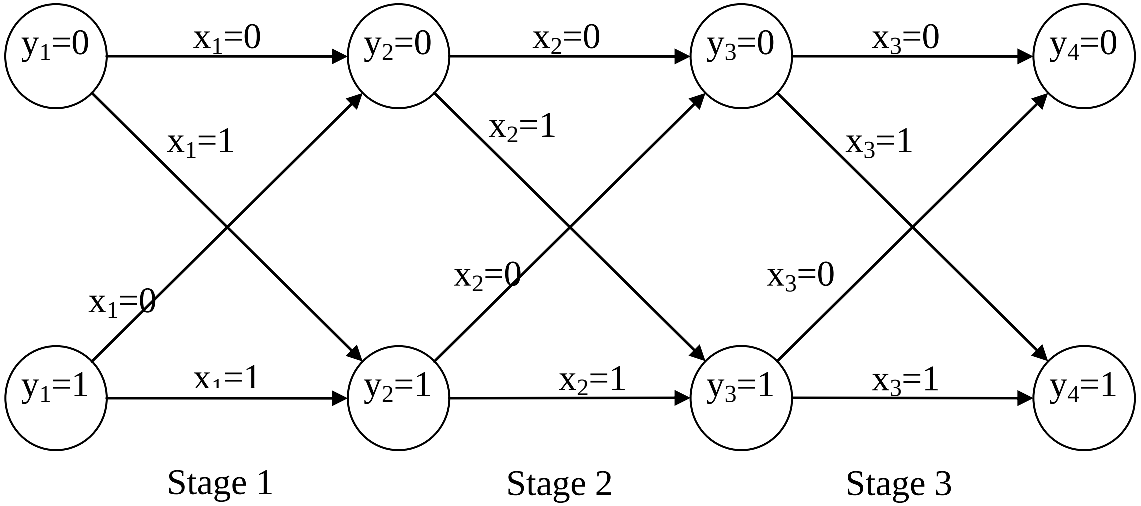
$$\mathbf{a}^{(i)} \text{ is equivalent to } \mathbf{a}^{(j)} \text{, iff} \quad \delta(\mathbf{a}^{(i)}) = \delta(\mathbf{a}^{(j)} \qquad (51)$$

The best multistage alternative $\mathbf{a}^{**}$ is defined as a multistage alternative which

- belongs to the non-empty class with the lowest index *m* and
- satisfies the relationship

$$\forall_{\mathbf{a}' \in \mathbf{A}^m} \ \delta(\mathbf{a}^{**}) \geq \delta(\mathbf{a}') \qquad (52)$$

5. Numerical illustration



$$\mathbf{A}_t = \{\mathbf{a}_t^{(0}, \mathbf{a}_t^{(1)}, \mathbf{a}_t^{(2)}, \mathbf{a}_t^{(3)}\}$$

$$\mathbf{a}_t^{(0)} = (0, 0), \quad \mathbf{a}_t^{(1)} = (0, 1) \quad \mathbf{a}_t^{(2)} = (1, 0) \quad \mathbf{a}_t^{(3)} = (1, 1).$$

At each stage we have two reference sets:

$$\mathbf{G}_t = \{\mathbf{g}_t^{(0)}, \mathbf{g}_t^{(1)}\} \text{ and } \mathbf{B}_t = \{\mathbf{b}_t^{(0)}, \mathbf{b}_t^{(1)}\}.$$

Values of the stage criteria weights

	c_t^1	c_t^2	c_t^3	c_t^4	c_t^5	c_t^6	c_t^7	c_t^8
t=1	0.17	0.23	0.23	0.12	0.25	0	0	0
t=2	0.17	0	0.23	0.12	0	0.28	0	0.2
t=3	0	0	0.23	0	0.25	0.27	0.11	0.14

Results of the comparisons of the stage alternatives with the elements of the reference sets

t	$\mathbf{A_t}$	$\mathbf{G_t}$	f_t^1	f_t^2	f_t^3	f_t^4	f_t^5	f_t^6	f_t^7	f_t^8	$\mathbf{B_t}$	f_t^1	f_t^2	f_t^3	f_t^4	f_t^5	f_t^6	f_t^7	f_t^8
1	$\mathbf{a}_1^{(0)}$	$\mathbf{g}_1^{(0)}$	–	=	–	+	=				$\mathbf{b}_1^{(0)}$	+	+	=	–	–			
		$\mathbf{g}_1^{(1)}$	–	–	=	–	=				$\mathbf{b}_1^{(1)}$	=	+	=	–	+			
	$\mathbf{a}_1^{(1)}$	$\mathbf{g}_1^{(0)}$	=	+	=	–	–				$\mathbf{b}_1^{(0)}$	+	=	–	=	+			
		$\mathbf{g}_1^{(1)}$	+	–	=	–	+				$\mathbf{b}_1^{(1)}$	=	=	+	+	=			
	$\mathbf{a}_1^{(2)}$	$\mathbf{g}_1^{(0)}$	–	–	–	=	=				$\mathbf{b}_1^{(0)}$	–	=	–	=	–			
		$\mathbf{g}_1^{(1)}$	–	–	=	–	+				$\mathbf{b}_1^{(1)}$	+	=	+	–	–			
	$\mathbf{a}_1^{(3)}$	$\mathbf{g}_1^{(0)}$	=	–	=	+	=				$\mathbf{b}_1^{(0)}$	–	=	=	–	+			
		$\mathbf{g}_1^{(1)}$	–	+	–	–	+				$\mathbf{b}_1^{(1)}$	+	+	–	+	–			
2	$\mathbf{a}_2^{(0)}$	$\mathbf{g}_2^{(0)}$	–		–	+		=	–		$\mathbf{b}_2^{(0)}$	+		+	=		–	+	
		$\mathbf{g}_2^{(1)}$	–		–	–		–	=		$\mathbf{b}_2^{(1)}$	–		+	=		+	–	
	$\mathbf{a}_2^{(1)}$	$\mathbf{g}_2^{(0)}$	–		=	=		–	=		$\mathbf{b}_2^{(0)}$	–		=	=		=	=	
		$\mathbf{g}_2^{(1)}$	=		=	–		=	=		$\mathbf{b}_2^{(1)}$	–		+	=		=	=	
	$\mathbf{a}_2^{(2)}$	$\mathbf{g}_2^{(0)}$	=		–	+		=	=		$\mathbf{b}_2^{(0)}$	+		+	+		–	+	
		$\mathbf{g}_2^{(1)}$	–		+	–		+	–		$\mathbf{b}_2^{(1)}$	=		–	+		=	+	
	$\mathbf{a}_2^{(3)}$	$\mathbf{g}_2^{(0)}$	–		–	=		+	=		$\mathbf{b}_2^{(0)}$	+		+	=		+	–	
		$\mathbf{g}_2^{(1)}$	–		–	=		+	–		$\mathbf{b}_2^{(1)}$	+		=	–		–	+	
3	$\mathbf{a}_3^{(0)}$	$\mathbf{g}_3^{(0)}$			–		–	+	=	–	$\mathbf{b}_3^{(0)}$			+		+	–	+	–
		$\mathbf{g}_3^{(1)}$			–		+	=	+	+	$\mathbf{b}_3^{(1)}$			+		+	–	=	+
	$\mathbf{a}_3^{(1)}$	$\mathbf{g}_3^{(0)}$			=		=	–	–	–	$\mathbf{b}_3^{(0)}$			–		–	+	–	=
		$\mathbf{g}_3^{(1)}$			–		–	–	=	+	$\mathbf{b}_3^{(1)}$			–		+	+	+	+
	$\mathbf{a}_3^{(2)}$	$\mathbf{g}_3^{(0)}$			–		+	–	–	–	$\mathbf{b}_3^{(0)}$			+		–	–	–	–
		$\mathbf{g}_3^{(1)}$			–		–	=	+	+	$\mathbf{b}_3^{(1)}$			–		+	–	+	–
	$\mathbf{a}_3^{(3)}$	$\mathbf{g}_3^{(0)}$			+		–	–	=	=	$\mathbf{b}_3^{(0)}$			+		+	=	–	–
		$\mathbf{g}_3^{(1)}$			–		+	=	–	–	$\mathbf{b}_3^{(1)}$			=		=	+	+	=

Results of the calculations of stage indicators.

Stage	$\mathbf{A_t}$	$d_{\mathbf{G}}^{+}(\mathbf{a}_t)$	$d_{\mathbf{G}}^{-}(\mathbf{a}_t)$	$d_{\mathbf{B}}^{+}(\mathbf{a}_t)$	$d_{\mathbf{B}}^{-}(\mathbf{a}_t)$
t=1	$\mathbf{a}_1^{(0)}$	0	0.88	0.63	0
	$\mathbf{a}_1^{(1)}$	0.65	0	0.77	0
	$\mathbf{a}_1^{(2)}$	0	0.75	0	0.64
	$\mathbf{a}_1^{(3)}$	0	0.52	0.52	0
t=2	$\mathbf{a}_2^{(0)}$	0	0.88	0.63	0
	$\mathbf{a}_2^{(1)}$	0	1	0	1
	$\mathbf{a}_2^{(2)}$	0.51	0	0.77	0
	$\mathbf{a}_2^{(3)}$	0	0.72	0	0.63
t=3	$\mathbf{a}_3^{(0)}$	0.77	0	0.59	0
	$\mathbf{a}_3^{(1)}$	0	0.61	0	0.73
	$\mathbf{a}_3^{(2)}$	0	0.75	0	0.77
	$\mathbf{a}_3^{(3)}$	0	0.75	0.75	0

Multistage Bipolar sorting and bipolar ranking

A	**A**$_1$, **A**$_2$, **A**$_3$	$d_{\mathbf{G}}^{+}(\mathbf{a})$	$d_{\mathbf{G}}^{-}(\mathbf{a})$	$d_{\mathbf{B}}^{+}(\mathbf{a})$	$d_{\mathbf{B}}^{-}(\mathbf{a})$	$d(\mathbf{a})$	$\mathbf{A}^i$	R
$\mathbf{a}^{(0)}$	$\mathbf{a}_1^{(0)}, \mathbf{a}_2^{(0)}, \mathbf{a}_3^{(0)}$	0.257	0.587	0.617	0	0.287	2	3
$\mathbf{a}^{(1)}$	$\mathbf{a}_1^{(0)}, \mathbf{a}_2^{(0)}, \mathbf{a}_3^{(1)}$	0	0.79	0.42	0.243	-0.61	5	10
$\mathbf{a}^{(2)}$	$\mathbf{a}_1^{(0)}, \mathbf{a}_2^{(1)}, \mathbf{a}_3^{(2)}$	0	0.877	0.21	0.59	-1.26	5	15
$\mathbf{a}^{(3)}$	$\mathbf{a}_1^{(0)}, \mathbf{a}_2^{(1)}, \mathbf{a}_3^{(3)}$	0	0.877	0.46	0.333	-0.75	5	11
$\mathbf{a}^{(4)}$	$\mathbf{a}_1^{(1)}, \mathbf{a}_2^{(2)}, \mathbf{a}_3^{(0)}$	0.643	0	0.71	0	1.353	1	1
$\mathbf{a}^{(5)}$	$\mathbf{a}_1^{(1)}, \mathbf{a}_2^{(2)}, \mathbf{a}_3^{(1)}$	0.387	0.203	0.513	0.243	0.453	3	4
$\mathbf{a}^{(6)}$	$\mathbf{a}_1^{(1)}, \mathbf{a}_2^{(3)}, \mathbf{a}_3^{(2)}$	0.217	0.49	0.257	0.467	-0.48	3	9
$\mathbf{a}^{(7)}$	$\mathbf{a}_1^{(1)}, \mathbf{a}_2^{(3)}, \mathbf{a}_3^{(3)}$	0.217	0.49	0.507	0.21	0.023	3	5
$\mathbf{a}^{(8)}$	$\mathbf{a}_1^{(2)}, \mathbf{a}_2^{(0)}, \mathbf{a}_3^{(0)}$	0.257	0.543	0.407	0.213	-0.09	3	7
$\mathbf{a}^{(9)}$	$\mathbf{a}_1^{(2)}, \mathbf{a}_2^{(0)}, \mathbf{a}_3^{(1)}$	0	0.747	0.21	0.457	-0.99	5	13
$\mathbf{a}^{(10)}$	$\mathbf{a}_1^{(2)}, \mathbf{a}_2^{(1)}, \mathbf{a}_3^{(2)}$	0	0.833	0	0.803	-1.64	6	16
$\mathbf{a}^{(11)}$	$\mathbf{a}_1^{(2)}, \mathbf{a}_2^{(1)}, \mathbf{a}_3^{(3)}$	0	0.833	0.25	0.547	-1.13	5	14
$\mathbf{a}^{(12)}$	$\mathbf{a}_1^{(3)}, \mathbf{a}_2^{(2)}, \mathbf{a}_3^{(0)}$	0.427	0.173	0.627	0	0.88	2	2
$\mathbf{a}^{(13)}$	$\mathbf{a}_1^{(3)}, \mathbf{a}_2^{(2)}, \mathbf{a}_3^{(1)}$	0.17	0.377	0.43	0.243	-0.02	3	6
$\mathbf{a}^{(14)}$	$\mathbf{a}_1^{(3)}, \mathbf{a}_2^{(3)}, \mathbf{a}_3^{(2)}$	0	0.663	0.173	0.467	-0.96	5	12
$\mathbf{a}^{(15)}$	$\mathbf{a}_1^{(3)}, \mathbf{a}_2^{(3)}, \mathbf{a}_3^{(3)}$	0	0.663	0.423	0.21	-0.45	5	8

6. Discussion

Comparison to to pure random sampling of the decision space.

Average values of $d_{\mathbf{G}}^{-}(\mathbf{a}_t)$, $d_{\mathbf{B}}^{+}(\mathbf{a}_t)$, $d_{\mathbf{B}}^{-}(\mathbf{a}_t)$

$$d_{\mathbf{G}}^{+avr},\ d_{\mathbf{G}}^{-avr},\ d_{\mathbf{B}}^{+avr},\ d_{\mathbf{B}}^{-avr}.$$

We have:

$$A^{avr} = 4.0.$$

$$d_{\mathbf{G}}^{+avr} = 0.161, \qquad d_{\mathbf{G}}^{-avr} = 0,572,$$

$$d_{\mathbf{B}}^{+avr} = 0.388, \qquad d_{\mathbf{B}}^{-avr} = 0.314.$$

We have:

$$d_{\mathbf{G}}^{+avr} < d_{\mathbf{G}}^{-avr} \tag{53}$$

$$d_{\mathbf{B}}^{+avr} > d_{\mathbf{B}}^{-avr} \tag{54}$$

$$d_{\mathbf{G}} = d_{\mathbf{G}}^{+avr} - d_{\mathbf{G}}^{-avr} = -\,0.411 \tag{55}$$

$$d_{\mathbf{B}} = d_{\mathbf{B}}^{+avr} - d_{\mathbf{B}}^{-avr} = 0.074 \tag{56}$$

$$d_{\mathbf{G}} < 0 \qquad d_{\mathbf{B}} > 0$$

$$d_{\mathbf{A}\cap\mathbf{B}} = d_{\mathbf{G}} + d_{\mathbf{B}} = -\,0.411 + 0.074 = -0.336 \tag{57}$$

7. Concluding remarks

I have made certain simplifying assumptions, which should be eliminated in the future. They include: setting the equivalence threshold at 0 and the concordance threshold at 0.5, as well as not using the veto coefficients. These are elements of the Electre methodology, used in the classic Bipolar method. Further research will aim at eliminating these limitations.

In the general case it may happen that certain stage alternatives are not comparable with the stage reference sets. In such situations, certain multistage alternatives will also be non-comparable.

One of the future directions of the extension of the method is the preparation of a general case description.

Another direction is to design software for numerical simulations. A further research direction would be to replace the enumerative method presented in this paper by methods based on multicriteria dynamic programming and genetic algorithms.

This procedure can be applied to create a long-term development strategy.

References

Ahmad M. (2018). Solution of multistage decision making problem through dynamic programming, *International Journal of Statistics and Applied Mathematics*, Vol. 3, Issue 2, Part E, 343-345.

Bellman R. (1957). Dynamic programming, Princeton University Press.

Dominiak C. (1997). Portfolio selection using the idea of reference solution. In: G. Fandel, T. Gal (eds.) *Multiple Criteria Decision Making*, Springer-Verlag Berlin Heidelberg New York, 593-602.

Górecka D. (2014). Metoda BIPOLAR z dominacjami stochastycznymi [in:] T. Trzaskalik (ed.), *Wielokryterialne wspomaganie decyzji. Metody i zastosowania*, PWE, Warszawa, 149-152.

Górecka D. (2017b). BIPOLAR MIX – a method for mixed evaluations and its application to the ranking of European projects, *Multiple Criteria Decision Making*, vol. 12 (2017).

Konarzewska-Gubała E. (1987). Multicriteria decision analysis with BIPOLAR reference system: Theoretical model and computer implementation, *Archiwum Automatyki i Telemechaniki*, vol. 32 no 4, 289-300.

Konarzewska-Gubała E. (1989). BIPOLAR: *Multiple Criteria Decision Aid Using Bipolar Reference System*, LAMSADE, Cahiers et Documents no 56, Paris.

Konarzewska-Gubała E. (1996). Supporting an effective performance appraisal system. *Argumenta Oeconomica* vol.1, 123-125.

Konarzewska-Gubała E. (2002). Multiple criteria company benchmarking using the BIPOLAR method. In: Trzaskalik T., Michnik J. (eds.) *Multiple Objective and Goal Programming. Recent Developments* Physica-Verlag. Springer-Verlag Company, Heidelberg, New York, 338-350.

Merighi D.(1980). Un modello di valutazione rispetto insiemi di riferimento assegnati. *Ricerca Operativa* no 13, 31-52.

Rapoport A. (1967). Dynamic programming models for multistage decision-making tasks. *Journal of Mathematical Psychology*, Vol. 4 no. 1, 48-71.

Roy B.(1985). *Méthodologie multicritère d'aide à la décision*. Economica, Paris.

Roy B, Bouyssou D. (1993). *Aide multicritère à la décision: Méthodes et cas*, Economica, Paris.

Tłuczak A. (2018). Reference points in the assessment of the decision variants – Bipolar method, *Quantitative methods in Economics*, Vol. XIX/3, 2018, 293–302 (in Polish).

Trzaskalik T. (1987). Model of multistage multicriteria decision processes applying reference sets. *Scientific Works of the University of Economics in Wroclaw* no. 413, 73-93 (in Polish).

Trzaskalik T. (1998). *Multiobjective analysis in dynamic environment,* University of Economics in Katowice.

Trzaskalik T., Sitarz S. (2012). How to deal with overgood and underbad alternatives in BIPOLAR method. In: Watada J., Watanabe T., Phillips-Wren G., Howlett R., Jain L.C. (eds.). *Proceedings of the 4th International Conference on Intelligent Decision Technologies (IDT´2012), Intelligent Decision Technologies Smart Innovation, Systems and Technologies* vol. 16, 345-354.

Trzaskalik T., Sitarz S., Dominiak C. (2013). Unified procedure for Bipolar method. In: Zadnik L. Żerovnik J., Povh J., Drobne S., Lisec A. (eds.). *The 12th International Symposium on Operational Research, Slovenian Society Informatika – Section for Operational Research*, 213-218.

Trzaskalik, T., Sitarz, S., Dominiak, C. (2019). Bipolar method and its modifications. *Cent Eur J Oper Res* **27,** 625–651.

Thank you
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